A DEFLATIONARY ACCOUNT OF THE TRUTH OF THE GÖDEL SENTENCE $\mathcal{G}$

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Abstract. We give a negative answer to the question of whether our conviction about the truth of the Gödel sentence $\mathcal{G}$ involves a theory of truth beyond the deflationary theories ([28],[21],[30],[22], [31],[6]). After discussing and dismissing Neil Tennant's deflationary account of incompleteness, we show how a new deflationary construal of the *incompletability* of formal systems can be framed in the setting of Peano Arithmetic augmented to include a *constructive* version of the ω-rule based on Herbrand's notion of prototype proof.

Keywords: Deflationism about truth · Incompleteness · Extensions of Peano Arithmetics · Reflection Principle · ω-rule · prototype proofs.

1. Introduction

According to deflationism, truth is a metaphysically thin property, redundant and dispensable, but useful as generalization device as in "whatever the oracle told you is true". Philosophers of course disagree whether deflationism about truth is true. Some objections to it are more penetrating than others, but none of them seems to be decisive. Yet some authors such as Stewart Shapiro and Jeffrey Ketland have argued that the refutation of deflationism can be, to some extent, a *mathematical* task: it is a matter of showing that our conviction about the truth of the independent Gödel sentence $\mathcal{G}$ involves "a theory of truth which *significantly transcends the deflationary theories*" ([21, p. 88], [28]). More specifically, Shapiro and Ketland maintain that a truth-theoretic extension of a given arithmetical formal system such as Peano Arithmetic PA is deflationarily licit only when it satisfies the *conservativeness* requirement, i.e. when "[it does] not allow us to prove anything in the original language that we could not prove before we added the truth predicate" [28, p. 497]. Thus, as $\mathcal{G}$ is independent of PA albeit expressed within its language – so the antideflationary argument goes – any truth-theoretic extension allowing us to prove the truth of $\mathcal{G}$ *must* be non-conservative and so deflationarily illicit.

This view has provoked Neil Tennant's reply on behalf of deflationism, according to which the Gödel sentence $\mathcal{G}$ can actually be recognised to be 'true', in the sense of 'assertable', without deploying or invoking a 'thick' concept of truth, i.e. avoiding the semantical notion of model or a Tarski's style truth-predicate. In particular, Tennant proposes a way of deflationarily achieving the proposition $\mathcal{G}$ by means of reflective extensions of formal arithmetic which augment the deductive apparatus of PA with a suitable version of the reflection principle ([30]). In this way, much current debate about deflationism and Gödel phenomena has been subsumed under the discussion about the justificatory status of reflective statements without appeal to non-conservative truth-theoretic extensions of PA ([22, 31, 6]).

Generally speaking, a natural way to be deflationist in mathematics is to equate truth with proof. However, the received view of incompleteness is spontaneously *inflationary* holding that the constitutive element of the First Incompleteness Theorem is the *indipendence* of truth from proof in PA, so that this theorem would prove the existence of arithmetical sentences which are *true* but *unprovable*. The current orthodoxy therefore favors an anti-deflationary stance: truth can be easily conceived of as a substantial, not a deflationary property of some sentences, if there is a discontinuity between their truth and their proof. As the anti-deflationist Ketland puts it – our understanding of the significance of the First Incompleteness Theorem is primarily a matter of sensitivity to the *proof-transcending* truth of $\mathcal{G}$ ([21, p. 91]).

Admittedly, the inflationary reading of incompleteness is as old as Kurt Gödel's own informal argument in the very first paragraph of his 1931 article. This argument, indeed, incorporates a commitment to a 'thick', primitive, concept of truth. Let PA denote the formal system of first order arithmetic and let

$$\mathcal{G} \Leftrightarrow \mathcal{G} \textit{ is not provable in } \mathsf{PA}$$

Let us now ask whether $\mathcal{G}$ is provable or not in PA.

- Suppose that $\mathcal{G}$ is provable in PA. Then, *for what it literally says of itself*, it is a false statement. This means that PA is unsound inasmuch as it allows a false statement to be proved. Hence: if PA is sound, then $\mathcal{G}$ is unprovable in it.
- Suppose instead that $\mathcal{G}$ is unprovable in PA. Then, it is a true statement and its negation $\neg\mathcal{G}$ is false. Again, if PA is sound, then $\neg\mathcal{G}$ is unprovable.

Therefore, PA is *syntactically incomplete*: there exists a statement $\mathcal{G}$ such that neither $\mathcal{G}$ nor its negation $\neg\mathcal{G}$ is provable in PA. Moreover, since $\mathcal{G}$ is a true statement, it follows that PA is also *semantically incomplete*, i.e. there exists a true statement that PA cannot prove [13].

Yet this semantical argument that Gödel launches as a sort of guide for the perplexed performs the function of an heuristic insight. (It should also be noted in passing that Gödel qualifies the independent statement as "richtig" not as "wahr"). The insight in question, engaging as it is, departs from the effective logical meaning of $\vdash_{\mathsf{PA}} \mathcal{G} \leftrightarrow \neg Theor_{\mathsf{PA}}(\ulcorner\mathcal{G}\urcorner)$, which states that $\mathcal{G}$ and $\neg Theor_{\mathsf{PA}}(\ulcorner\mathcal{G}\urcorner)$ are mutually interchangeable with regard to provability within PA. So the gap between $\mathcal{G}$ and its supposed translation in terms of natural language cannot by any means be filled by a precise logical argument. However, in the sequel of his article Gödel sets the scene for incompleteness in purely syntactical (and moreover intuitionistic) terms so that the core of his construction properly involves a syntactical sensitivity rather than a semantical one.

On the other hand, the fact that proof and arithmetical truth in PA do not co-travel should be viewed only as a *consequence* of the First Incompleteness Theorem *under a classical view of truth*: insofar as $\nvdash_{\mathsf{PA}} \mathcal{G}$ and $\nvdash_{\mathsf{PA}} \neg\mathcal{G}$, one grants that either $\mathcal{G}$ or $\neg\mathcal{G}$ must be true. But the *irrelevance* of bivalence from a *mathematical* point of view suggests a rationale for a deflationary approach to incompleteness. This point can be illustrated by means of an example. Consider Σ_1-completeness in its contraposed form: since $\neg\mathcal{G}$ is a Σ_1-statement such that $\nvdash_{\mathsf{PA}} \neg\mathcal{G}$, then $\neg\mathcal{G}$ is false in the standard model $\mathcal{N}$ and, therefore, $\mathcal{G}$ is true in $\mathcal{N}$ (see the Corollary A.9). In other words, the Σ_1-completeness allows us to achive the truth of $\mathcal{G}$ in the standard model by the very independence of $\mathcal{G}$ from PA. Now take Goldbach's conjecture: *for all $n \in \mathbb{N}$, if n is greater than 2, then it can be expressed as the sum of two primes.* Like $\mathcal{G}$, Goldbach's conjecture is a Π_1-statement. Let $\pi(x)$ be the predicate of being a prime number, the conjecture can actually be formalised as follows:

$$(\mathsf{GC})\quad \forall x(x > 2 \to \exists y < x \exists z < x(x = y + z \wedge \pi(y) \wedge \pi(z))).$$

Imagine then that someone were to prove that GC is independent from PA and so, by an argument analogous to that for $\mathcal{G}$, that GC is true in the standard model $\mathcal{N}$; again, the truth of GC is yielded by its independence proof. We think it is safe to notice that mainstream number theorists would feel inclined to seek a *counterexample* of Goldbach's statement in elementary number theory (or a proof of it in higher systems).

Another source of epistemological worries about inflationary (model-theoretical) demonstrations of $\mathcal{N} \vDash \mathcal{G}$ concerns the inegalitarianism about models, that is the choice of the standard model $\mathcal{N}$ as the official platform for establishing the truth of $\mathcal{G}$. A general line of reasoning runs as follows.

(1) The central aim of a formal system of arithmetic is to give a formal account for elementary number theory as faithful as possible.
(2) But any model of the expanded theory $\mathsf{PA} \cup \{\neg\mathcal{G}\}$ (i.e. the theory where $\neg\mathcal{G}$ is assumed to be true) must be a *non-standard* one.
(3) Therefore, the exclusion of the standard model $\mathcal{N}$ from the range of possible mathematical structures verifying the axioms of the theory would flout our intuition, because $\mathcal{N}$ expresses the *intended* structure of natural numbers, i.e. it does not include heterogeneous entities like non-standard numbers.

As Michael Dummett points out in his famous paper on Gödel's Theorem, this argument is epistemologically plagued by the puzzling loop caused by the notion of standard model [8]. In Crispin Wright's words, "as soon as it is granted that any intuitively sound system of arithmetic *merely* partially describes the subject matter to which it answers, an explanation is owing of *how* the subject matter in question can possess a determinacy transcending complete description" [34]. In practice, the intended structure of elementary number theory $\mathcal{N}$ is the theory we want to show as being free from contradictions and to this end we try to characterise faithfully the class of its truths by means of the property of being a theorem of PA. But our judgement that $\mathcal{G}$ is true though unprovable in PA depends for its acceptability on the *assumption* that the structure $\mathcal{N}$ is clear enough to regulate the deductive behaviour of PA.

In this paper, we articulate a new deflationary construal of incompleteness where the concept of truth has no substantial role to play in our convinction that the independent sentence $\mathcal{G}$ should be asserted. Moreover, we focus on the actual conceptual core of the Gödelian construction, namely the *deductive inexhaustibility* of PA. Indeed, whereas inflationary theories of truth tends to 'complete' PA while proving $\mathcal{G}$, current deflationary accounts dismiss the problem of the incompletability of formal systems. Our syntactical path leading to the achievement of $\mathcal{G}$ avoids the heavy commitment to reflection principles, so transparently *ad hoc*. As a consequence, we deny the thesis about the mathematical refutability of deflationism about truth via Gödel's Theorem, while bypassing as inessential the anti-deflationary demand for the derivability of the reflection principles without truth-theoretic principles [10].

The plan of the paper is as follows. In the next Section, we discuss a whole set of problems emerging from the deflationary approach to incompleteness proposed by Tennant in [30]. In Section 3, we indicate a syntactical, deflationary route to $\mathcal{G}$. In particular, we pursue a reading of incompleteness in terms of the (constructive) ω-rule and the notion of *prototype proof* in Jacques Herbrand's sense of the term. The constructive ω-rule is shown to have a deflationary character, so as not only to avoid (necessarily non-conservative) semantical justifications, but also to overcome the very inexhaustibility phenomenon. Finally, we briefly discuss our deflationary proposal in relation to the procedure sketched by Dummett for achieving the truth of $\mathcal{G}$. In order to make the paper as self-contained and readable as possible, the appendix provides notations and basic notions as well as the proofs of the theorems involved.

2. Against Tennant's deflationary reading of incompleteness

In order to provide a proof for $\mathcal{G}$ in an augmented formal theory including PA, Tennant suggests extending PA with a reflection principle in Feferman's spirit [9] – i.e. an axiom-schema that disquotes the truth-predication coming from theoremhood. Under this principle, called the *principle of uniform primitive recursive reflection*, he intends to show "that there is a 'deflationary way' of faithfully carrying out the semantical argument for the truth of the independent Gödel sentence" [30, p. 557].

For ease of exposition, we sketch his argument by taking into account the formal system PA^{Rfn}, that is PA expanded to include the *local reflection principle*:

$$Rfn\text{: } \forall \alpha, Theor_{\mathsf{PA}}(\overline{\ulcorner \alpha \urcorner}) \to \alpha.$$

PA^{Rfn} is called the *soundness extension* of PA for the reason that the *Rfn* principle is taken into account to represent the formal counterpart to the metalogical property of soundness. Now, from the soundness of PA with regard to the standard model $\mathcal{N}$, follows in particular a proof of $\mathcal{N} \vDash \mathcal{G}$, so that it is possible to formalise a PA^{Rfn} proof of $\mathcal{G}$. We clearly have both

$$\vdash_{\mathsf{PA}^{Rfn}} Theor_{\mathsf{PA}}(\overline{\ulcorner \mathcal{G} \urcorner}) \to \mathcal{G}$$

and

$$\vdash_{\mathsf{PA}^{Rfn}} \neg Theor_{\mathsf{PA}}(\overline{\ulcorner \mathcal{G} \urcorner}) \to \mathcal{G}.$$

Therefore, by stressing the classical tautology

$$((\alpha \to \beta) \wedge (\neg\alpha \to \beta)) \to \beta$$

we can easily conclude

$$\vdash_{\mathsf{PA}^{Rfn}} \mathcal{G}.$$

The following three questions introduce as many objections that can be brought against Tennant's deflationism:

(1) *Does the reflection principle actually express the soundness property?*
(2) *Does the consistency extension lack good philosophical motivations?*
(3) *Does the reflection principle enable us to fill the gap between provability and truth?*

Let us approach these questions in turn.

2.1. **Does the reflection principle actually express the soundness property?** Tennant's argument depends on the assumption that the *Rfn* axiom schema expresses a *syntactical*, and so genuinely deflationary, rendition of the metatheoretical property of soundness within PA. Such a translation is meant to preserve at least the intensional meaning of the soundness property, that is the fact that if α is a theorem of PA, then α can be *accepted* as an arithmetical truth; this allows us to derive the epistemological justification for it from the very belief in the soundness of the theory. Now, the soundness of PA with respect to the standard model yields a model-theoretical proof of the *inflationary* truth of $\mathcal{G}$ [8]. Analogously, Tennant claims that a deflationary rendition of the soundness property will allow the achievement of a proof of $\mathcal{G}$ in a *deflationary* way.

But a fundamental difficulty comes into view by acknowledging that PA is sufficiently strong to prove the so-called *provable* Σ_1*-completeness* (see [26]), i.e. the fact that:

$$\forall\alpha \in \Sigma_1, \vdash_{\mathsf{PA}} \alpha \rightarrow Theor_{\mathsf{PA}}(\ulcorner\alpha\urcorner).^{1}$$

Σ_1-completeness brings with it the fact that any independent Π_1-formula is recognised to be true by $\mathcal{N}$. Specifically, it is possible to provide an easy proof for $\mathcal{N} \vDash \mathcal{G}$. Indeed, $\mathcal{G} \in \Pi_1$ and consequently $\neg\mathcal{G} \in \Sigma_1$; hence, by the contrapositive of the Σ_1-completeness, we obtain $\mathcal{N} \nvDash \neg\mathcal{G}$, i.e. $\mathcal{N} \vDash \mathcal{G}$. Accordingly, such a proof undermines the above defense of deflationism. Indeed, if the provable Σ_1-completeness actually represented its metatheoretical counterpart, then the sentence $\mathcal{G}$ would be provable within PA itself! So, the fact that the provable Σ_1-completeness can be understood on a disquotational parallel with the *Rfn* principle erases any trust in the intensional character of the correspondence between *Rfn* and the soundness property.

2.2. **Does the consistency extension lack good philosophical motivations?** Tennant quickly dismisses the *consistency extension* of PA, obtained by adding to PA axioms the sentence asserting the syntactical consistency of PA, $Cons_{\mathsf{PA}}$. The motivation for this dismissal lurks in the cryptic metaphor that consistency extension is "an uninformative hammer with which to crack the independent walnut" [30, p. 573]. But it is hard to make sense of this from the very perspective he embraces. First of all, as Dummett points out in his contribution on the Gödel's Theorem [8], the truth of $\mathcal{G}$ is a truth *under the assumption of the consistency of* PA. Now, it is well-known that through the Second Incompleteness Theorem this dependency can be strengthened and formalised within PA, $\mathcal{G}$ and $Consa_{\mathsf{PA}}$ being two provably equivalent propositions: $\vdash_{\mathsf{PA}} Cons_{\mathsf{PA}} \leftrightarrow \mathcal{G}$. In this way, a proof of $\mathcal{G}$ can be very simply obtained as the result of a *modus ponens* with the new axiom $Cons_{\mathsf{PA}}$. Secondly, the consistency extension takes advantage of being *weaker* than the soundness extension. In general, the soundness of an arithmetical theory T implies its consistency (otherwise T would prove false statements), but the converse does not hold given the existence of unsound consistent theories [17]. Indeed, since $Cons_{\mathsf{PA}}$ is

[1] Note that this formulation can be depurated from any residual metatheoretical feature, simply by individuating a recursive predicate $\mathfrak{S}(x)$ such that $\vdash_{\mathsf{PA}} \mathfrak{S}(\overline{n})$ if and only if n is the Gödelian coding of a Σ_1-formula. In such a way, the provable Σ_1-completeness turns to be condensed into the following axiom-schema:

$$\forall\alpha, \vdash_{\mathsf{PA}} \mathfrak{S}(\ulcorner\alpha\urcorner) \rightarrow (\alpha \rightarrow Theor_{\mathsf{PA}}(\ulcorner\alpha\urcorner)).$$

provably equivalent to $\mathcal{G}$, $\mathsf{PA} \cup \{Cons_{\mathsf{PA}}\}$ is a *minimal* deductive extension allowing us to prove $\mathcal{G}$ (similarly to Tennant's extension based on the *uniform primitive recursive reflection*). From an epistemological standpoint, belief in the soundness of our arithmetical theory implies belief in its consistency to the extent that the cognitive act of recognising a certain cluster of axioms as intuitively true runs under the implicit assumption of their reciprocal consistency. Could anyone recognise as evidently true a pair of axioms contradicting each other? In this way, belief in consistency turns out to be the very first step towards belief in the soundness of the theory. This is the real reason why the proof of the soundness of PA is regarded as uninformative with respect to the consistency of the system: it cannot prove a property silently assumed by the proof itself. To conclude, the point is not the lack of *good* reasons for setting aside the consistency extension *in favour* of the local reflection principle. Rather we say that an allegation against the consistency extension cannot be justified without compromising the acceptance of the soundness extension itself.

2.3. **Does the reflection principle enable us to fill the gap between provability and truth?** The First Incompleteness Theorem establishes much more than the syntactical incompleteness of PA: it shows that *any* first order formal system capable of faithfully representing a certain amount of elementary number theory is deductively *inexhaustible*. In the specific case of PA, if we attempt to fill the deductive hole $\mathcal{G}$ by extending its deductive power, the deductive hole replicates itself through a new independent proposition $\mathcal{G}'$. Such a phenomenon is due to the fact that the undecidable proposition $\mathcal{G}$ constructed by Gödel involves the predicate $Theor_{\mathsf{PA}}(x)$ which is, by definition, strictly dependent on the set of axioms of PA, so that it gives rise to an independence phenomenon which is intrinsically insurmountable[2].

In the case of the specific extension PA^{Rfn} studied by Solomon Feferman and avocated by Tennant, we can iterate Gödel's construction so as to produce another independent proposition $\mathcal{G}'$ such that $\vdash_{\mathsf{PA}^{Rfn}} \mathcal{G}' \leftrightarrow \neg Theor_{\mathsf{PA}^{Rfn}}(\ulcorner \mathcal{G}' \urcorner)$ and $\mathcal{G}' \neq \mathcal{G}$. Now there is surely something pretty dubious about the relation between Tennant's deflationary proposal and the inexhaustibility phenomenon. The soundness extension, of course, is a recipe for the incompleteness of any formal system; this means that, as we decide $\mathcal{G}$ by expanding PA to include the soundness principle $Theor_{\mathsf{PA}}(\ulcorner \alpha \urcorner) \rightarrow \alpha$, we can in turn expand PA^{Rfn} with the principle $Theor_{\mathsf{PA}^{Rfn}}(\ulcorner \alpha \urcorner) \rightarrow \alpha$ in order to decide $\mathcal{G}'$, and so on. But now the question raises an epistemologically subtle point: the extension which allows us to fill the present deductive hole, is the very *cause* of replication of the deductive hole itself. Thus, as soon as we decide $\mathcal{G}$, the achievement of its truth proves a mere fig leaf because *the process itself of deciding* $\mathcal{G}$ launches the question of deflationarily achieving the truth of the new independent proposition $\mathcal{G}'$.

From an epistemological standpoint, the situation would be radically different if at each deductive extension step we decided one (or finitely many) of the infinitely many independent propositions $\mathcal{G}_1, \mathcal{G}_2, \ldots$, so as to avoid a new deductive hole. Although this would be another case of deductive inexhaustibility, the mechanism of filling the deductive holes step by step would not give real cause for concern. On the contrary, the relation

[2] It is worth recalling here that the notion of "deductive inexhaustibility" can be fully characterized in plain recursion theoretic terms by means of both the well-known notions of *creative set* and *productive set* [27]. Such a kind of characterization turns out to be completely independent of classical model theory.

between Gödelian incompleteness and Tennant's deflationary strategy reenacts the sort of regress exposed in Zeno's paradox of Achilles and the tortoise: decidability will always remain behind with respect to the independent proposition which is under focus at each step.

To sum up: the real lesson of the case is that we must be careful not to take heuristic insights too seriously from a logical point of view: what the *Rfn* principle states is nothing but the fact that a proof of $\vdash_{\mathsf{PA}^{Rfn}} Theor_{\mathsf{PA}}(\overline{\ulcorner\alpha\urcorner})$ makes it possible a proof of $\vdash_{\mathsf{PA}^{Rfn}} \alpha$ by means of a *modus ponens* application between $\vdash_{\mathsf{PA}^{Rfn}} Theor_{\mathsf{PA}}(\overline{\ulcorner\alpha\urcorner})$ and *Rfn* instantiated with α. On this view, Tennant's strategy for achieving the truth of $\mathcal{G}$ happens to be surprisingly close to the naive argument outlined in Section 1 whereby the sentence $\mathcal{G}$ is true for it declares that 'I'm unprovable' and in fact it is unprovable. However both the soundness and the consistency extensions are at odds with the need to stem new independent propositions. This is to conclude that any reliable approach to the problem of deflationarily recognising $\mathcal{G}$ as a true statement has to focus attention on the main concern of Gödel's work, which is the inexhaustibility of formal arithmetic. This is what we will do in the next section.

3. An alternative deflationary proposal: the constructive ω-rule

3.1. **The unrestricted ω-rule.**

The shift from PA to the so-called ω-logic (henceforth indicated by PA^{ω}) gives us our starting point for proving $\mathcal{G}$, while avoiding the surfacing of another independent proposition. Let us recall that PA^{ω} is obtained from PA by adding the ω-rule in place of the rule of induction: we can infer that $\forall x\alpha(x)$, provided we can prove $\alpha(n)$ for *each* natural number n. Formally:

$$\frac{\vdash_{\mathsf{PA}^{\omega}} \alpha(\overline{0}) \quad \vdash_{\mathsf{PA}^{\omega}} \alpha(\overline{1}) \quad \vdash_{\mathsf{PA}^{\omega}} \alpha(\overline{2}) \quad \dots}{\vdash_{\mathsf{PA}^{\omega}} \forall x\alpha(x)}\ \omega\text{-rule.}$$

The above rule was first described in a published work by David Hilbert in 1931 [15][3]. PA^{ω} does not only decide both the Gödelian propositions $\mathcal{G}$ and $Cons_{\mathsf{PA}}$, but has the capacity for providing a *syntactically complete* characterisation of first order arithmetic (see Corollary A.11).

The inclusion of a new inference rule has the advantage over the axiomatic extensions of allowing us to avoid any cumbersome philosophical commitment on the capacity of a certain axiom schema to syntactically reproduce a metatheorical property. From an epistemological point of view, moreover, the ω-rule needs no particular justification: the inferential device it expresses is largely supported by our intuition about the structure of natural numbers so as to be straightforwardly sound when one refers to the standard model.

Yet, of course, this route to $\mathcal{G}$ via the infinitely many premisses of the ω-rule corresponds to the collapse of the Hilbertian notion of formal system, and this fact is just another way of stating the First Incompleteness Theorem. The best we can say is that PA^{ω} makes a virtue out of the very ineffectiveness of the notion of proof by circumventing the phenomenon of deductive inexhaustibility of formal arithmetic, given that the predicate

[3] For an accurate history of the ω-rule see [16].

$Theor_{\mathsf{PA}}(x)$ cannot be upgraded to $Theor_{\mathsf{PA}^\omega}(x)$. Anyway, we are driven back to a notion of truth which is *symmetric* with respect to the thin version sponsored by the advocate of deflationism: a thick absolute as the upshot of an infinitary non-constructive reasoning. In effect, why not say that the movement from PA to PA^ω is anything but a *semantical* transition? The infinitary nature of ω-rule suggests that the rule intervenes as an external device to stretch syntactically the Tarskian definition of truth for the universal quantifier within arithmetical theories:

$$\forall x\alpha(x) \textit{ is true if, and only if, } \alpha(n) \textit{ is true for all } n \in \mathbb{N}.$$

This is why PA^ω succeeds in achieving completeness (see the proof of Theorem A.10), while the semantical completeness of PA with respect to the standard model $\mathcal{N}$ does not exceed the level of Σ_1-formulas.

The present situation is puzzling for a deflationary approach to incompleteness: on the one hand, the First Incompleteness Theorem pulls any epistemologically well-founded attempt to decide $\mathcal{G}$ away from the notion of formal system; on the other hand, any deductive strategy involving non-formalisable devices like the ω-rule is a *semantical* strategy in disguise. To be more precise, the question now before us is whether there is room between formal arithmetic and its classical semantics for a genuine syntactical manoeuvre able to achieve the provability of $\mathcal{G}$.

3.2. **The constructive ω-rule.** A path for an affirmative answer to this question is provided by the *constructive* ω-rule:

$$\text{if } \alpha(\overline{n}) \text{ admits a } \textit{prototype} \text{ proof w.r.t. } n \in \mathbb{N}, \text{ then conclude } \vdash_{\mathsf{PA}} \forall x\alpha(x).$$

Following Michael Detlefsen [7], the term 'prototype' assumes the meaning attached to it by Jacques Herbrand in "Sur la non-contradiction de l'Arithmetique" (1931): "when we say that a theorem is true for all x, we mean that for each x individually it is possible to iterate its proof, which may just be considered a prototype of each individual proof"[14]. In other words, a prototype proof provides "a reasoning which uniformly holds for all arguments, and this uniformity allows (and it is guaranteed by) the use of a generic argument" [24].

Alan Bundy and his co-workers regards the constructive ω-rule as a device for capturing the notion of *schematic* proof:

> The constructive ω-rule is a refinement of the ω-rule that can be used in practical proofs. It has the additional requirement that the $\varphi(n)$ premises be proved in a *uniform* way, i.e. that there exists a recursive program, $\texttt{proof}_\varphi$, which takes a natural number n as input and returns a proof of $\varphi(n)$ as output. [...] The recursive program $\texttt{proof}_\varphi$ formalises our notion of schematic proof [4].[4]

[4] This definition tends to blur the distinction between the *constructive* and the *recursive* versions of the ω-rule. Following Shoenfield [29], Torkel Franzén writes: "A proof of ϕ in a system incorporating the recursive ω-rule is either a pair $\langle\phi, 0\rangle$ where ϕ is an axiom, or a sequence $\langle\phi, e_1, ..., e_n\rangle$ where e_i is a proof of ψ_i, and ϕ follows from $\psi_1, \ldots, \psi_n$ by some ordinary inference rule, or, if ϕ is $\forall x\psi$, a pair $\langle\phi, e\rangle$, where e is the index of a total recursive function such that $\{e\}(n)$ is a proof of $\psi(\overline{n})$ for every n" [12]. Indeed, the cleavage between the two versions is merely epistemological since the implementation of the constructive ω-rule requires the specification of a recursive function able to return a proof of $\psi(\overline{n})$ for every $n \in \mathbb{N}$ in input.

Perhaps the most articulated mathematical approach to prototype proofs has been devised by Giuseppe Longo with regard to impredicative type theory [23]. As far as the epistemological nature of the constructive ω-rule is concerned, he observes:

> [...] the proof of a universally quantified statement is not understood by following the naif (Tarskian style) interpretation of "$\forall x$..." as "for all x ...": in no way "$\forall x$..." is used in a proof in the sense of the inspection of "all instances" in the intended model, yet its meaning and use refer to x as generic in a prototype proof [23].

Henceforth, we indicate by $\omega_{\downarrow}$ and $\mathsf{PA}^{\omega_{\downarrow}}$, respectively, the constructive version of the ω-rule and the deductive system obtained by weakening PA^{ω} through the replacement of the unrestricted ω-rule with $\omega_{\downarrow}$ and the consequent reintroduction of the induction principle. Both proofs of $\vdash_{\mathsf{PA}^{\omega}} \mathcal{G}$ and $\vdash_{\mathsf{PA}^{\omega}} Cons_{\mathsf{PA}}$ are based on prototype arguments. Let us produce, for instance, the proof of $\vdash_{\mathsf{PA}^{\omega_{\downarrow}}} \mathcal{G}$. Its crucial prototype juncture is displayed in detail below:

1.	$\vdash_{\mathsf{PA}} Dem_{\mathsf{PA}}(\overline{n}, \overline{\ulcorner \mathcal{G} \urcorner})$	hypotheses by absurd
2.	$\vdash_{\mathsf{PA}} \exists x Dem_{\mathsf{PA}}(x, \overline{\ulcorner \mathcal{G} \urcorner})$	$\exists$-introduction
3.	$\vdash_{\mathsf{PA}} Theor_{\mathsf{PA}}(\overline{\ulcorner \mathcal{G} \urcorner})$	definition of $Theor_{\mathsf{PA}}(x)$
4.	$\vdash_{\mathsf{PA}} \neg \mathcal{G}$	Diagonalisation Lemma
5.	$\nvdash_{\mathsf{PA}} \neg \mathcal{G}$	First Incompleteness Theorem
6.	$\nvdash_{\mathsf{PA}} Dem_{\mathsf{PA}}(\overline{n}, \overline{\ulcorner \mathcal{G} \urcorner})$	absurd from 4,5
7.	$\vdash_{\mathsf{PA}} \neg Dem_{\mathsf{PA}}(\overline{n}, \overline{\ulcorner \mathcal{G} \urcorner})$	Δ_0-decidability.

Clearly, for any $m \in \mathbb{N}$ this argument allows the generation of a proof of $\vdash_{\mathsf{PA}} \neg Dem_{\mathsf{PA}}(\overline{m}, \overline{\ulcorner \mathcal{G} \urcorner})$ just by replacing $\overline{n}$ with $\overline{m}$. As PA is a subsystem of $\mathsf{PA}^{\omega_{\downarrow}}$, we straightforwardly have, for any $m \in \mathbb{N}$, that $\vdash_{\mathsf{PA}^{\omega_{\downarrow}}} \neg Dem_{\mathsf{PA}}(\overline{m}, \overline{\ulcorner \mathcal{G} \urcorner})$; finally, by a step of the $\omega_{\downarrow}$-rule, we can conclude that $\vdash_{\mathsf{PA}^{\omega_{\downarrow}}} \forall x \neg Dem_{\mathsf{PA}}(x, \overline{\ulcorner \mathcal{G} \urcorner}) \equiv \mathcal{G}$. Along similar lines, we can obtain a proof of $\vdash_{\mathsf{PA}^{\omega_{\downarrow}}} Cons_{\mathsf{PA}}$ by the Second Incompleteness Theorem.[5]

The epistemological cleavage produced through this process of 'constructivisation' is remarkable. The epistemological dividend that this process can pay may be fully appreciated when it is realised that the unrestricted ω-rule can be viewed as a sort of general pattern from which one can specify some different constructive versions. In this respect, the induction principle may be conceived of as a *specific* constructive instance of the ω-rule, where the infinitely many premisses for the universal quantification are generated by a well-defined recursive function which consists in the proof by induction itself. On the other hand, the infinite premisses of $\omega_{\downarrow}$ are generated by a certain prototype proof through successive replacements[6].

Both these constructive versions succeed in capturing a widespread pattern of reasoning in mainstream number theory, even in the most radically constructive contexts. As

[5] In general, the provability of the Gödelian propositions is due to the fact that the enriched theory $\mathsf{PA}^{\omega_{\downarrow}}$ enjoys the following additional derivability condition:

$$\mathscr{D}_{\omega}: \quad \textit{for any formula } \alpha, \nvdash_{\mathsf{PA}^{\omega_{\downarrow}}} \alpha \Rightarrow \vdash_{\mathsf{PA}^{\omega_{\downarrow}}} \neg Theor_{\mathsf{PA}}(\overline{\ulcorner \alpha \urcorner}).$$

Clearly, $\mathscr{D}_{\omega}$ does not hold true in PA, otherwise $\mathcal{G}$ would be provable and unprovable at the same time. As far as the validity of $\mathscr{D}_{\omega}$ in $\mathsf{PA}^{\omega_{\downarrow}}$ is concerned, the reader can find all the technical details by looking at Theorem A.12 and Corollary A.13.

[6] Specific implementations of constructive $\omega_{\downarrow}$-rule, especially in view of automatic deduction treatments, are afforded in [3, 4].

regards the $\omega_{\downarrow}$-rule, it is worth mentioning a family of diagrammatic proofs of basic arithmetical facts [18]. But, whereas the inductive mechanism can be compressed into a single axiom of a formal system like PA, the notion of 'prototypicality' seems to present intrinsic intensional features far from being formally reproducible [24].

The $\omega_{\downarrow}$-rule radically diverges from its unrestricted ancestor in the very logical way it introduces the universal quantifier. When the unrestricted ω-rule is applied, the inferential step leading from the infinitely many premisses $\alpha(0), \alpha(1), \alpha(2), \dots$ to the conclusion $\forall x \alpha(x)$ actually expresses, we might say, a *synthetic* inference inasmuch it abridges the infinity of its premisses into a finite syntactical expression. On the contrary, the constructive ω-rule introduces the universal quantifier in an *analytical* way, for the infinitary information about premisses is already finitarily encompassed into the logical structure of the prototype argument. This means that it is the *whole* prototype argument that accomplishes the synthetic task of enclosing the infinite into a finite demonstrative device. This the reason why $\omega_{\downarrow}$-rule has a double logical nature: finitary and non-formalisable at the same time.

An immediate consequence of such an epistemological reversal is that, unlike the unrestricted pattern of inference conveyed by the ω-rule, $\omega_{\downarrow}$ does not lie within the province of classical semantics. In fact, in the the process of constructive specification leading to $\omega_{\downarrow}$, the ω-rule loses its unconditional generality, so that it can no longer faithfully reproduce the Tarskian definition for the universal quantifier. In epistemological terms, the constructive requirement divorces the $\omega_{\downarrow}$-rule from the non-constructive semantics grounded on bivalence.

The notion of uniformity underlying the constructive ω-rule clearly recalls the first-order logical principle of Universal Generalization (GU) which licenses the inference from $\varphi(t)$ to $\forall x \varphi(x)$ provided the absolute *genericity* of the term t (technically, x does not appear as a free variable in $A(t)$ and $A(x)$ is the result of replacing all occurrences of t in $A(t)$ by x). Of course GU and $\omega_{\downarrow}$ cannot coincide, otherwise $\mathcal{G}$ would be provable against the First Incompleteness Theorem. Since their difference lies in the fact that $\omega_{\downarrow}$ is restricted to natural numbers, one might object that the adoption of $\omega_{\downarrow}$ entails a strong semantical commitment to them. Our reply is that a prototype proof do *not* presuppose the set of natural numbers, but *characterizes* it as the set of all the numerical entities to which the schematic argument at issue applies. In other words, when a prototype argument is turned into a universal quantification, the quantifier is meant to range over all the numerical objects capable of instantiating the argumentative schema. So, we may say that any prototype argument implictly *defines* a set of numbers. This aspect highlights another important difference with the unrestricted ω-rule which actually enjoins us to assume a numerical ontology in order to achieve to whole set of its premisses. Once that the relation between semantics and syntax has been reversed, the constructive ω-rule can save its deflationary skill[7].

[7]For similar reasons, this rule cannot be expressed by means of the *Uniform Reflection Scheme*:

$$Urs\colon \forall x Theor_{\mathsf{PA}}(\overline{\ulcorner \alpha(x) \urcorner}) \to \forall x \alpha(x).$$

Indeed, PA^{Urs} — i.e. PA with Urs added as a new axiom — is still an incomplete formal system by the First Incompleteness Theorem, whereas $\mathsf{PA}^{\omega_{\downarrow}}$ is syntactically complete. This fact, of course, cannot be avoided when our principle is formulated as a rule: *if* $\vdash_{\mathsf{PA}} \forall x Theor_{\mathsf{PA}}(\overline{\ulcorner \alpha(x) \urcorner})$, *then* $\vdash_{\mathsf{PA}} \forall x \alpha(x)$.

3.3. **Deconstructing Dummett's argument.** To test our deflationary proposal, let us accept the challenge raised by Tennant when he claims that "in so much as stating the philosophical crux of Gödel's theorem, Dummett has furnished the kind of use of the truth-predicate that any deflationist would wish to deconstruct" ([30, p.552]).

In 1963, Dummett sketched a procedure for achieving the truth of $\mathcal{G}$:

> The argument for the truth of [$\mathcal{G}$] proceeds under the hypothesis that the formal system in question is consistent. The system is assumed, further, to be such that, for any decidable predicate $B(x)$ and any numeral $\overline{n}$, $B(\overline{n})$ is provable if it is true, — $\neg B(\overline{n})$ is provable if $B(\overline{n})$ is false (the notions of truth and falsity for such a statement being, of course, unproblematic). The particular predicate $A(x)$ [i.e., $\neg Dem(x, \ulcorner \mathcal{G} \urcorner)$] is such that, if $A(\overline{n})$ is false for some numeral $\overline{n}$, then we can construct a proof in the system of $\forall x A(x)$. From this it follows — on the hypothesis that the system is consistent — that each of $A(0)$, $A(1)$, $A(2)$, ...is true'. [8, p. 192]

As is so often the case, the fortune of a certain idea profits from a certain vagueness in its formulation, so that many authors have tried to make Dummett's hint more specific. However, it is rather surprising that it has never been detected here the offices of a prototype argument. It is because Dummett's argument has a prototypical nature, indeed, that the shift from PA to $\mathsf{PA}^{\omega_\downarrow}$ yields a deflationary reduction of it. Specifically, two logical circumstances make this reduction possible. The first refers to the 'analytical' way in which the $\omega_\downarrow$-rule introduces the universal quantifier. This epistemological feature matches Dummett's claim that "[...] the transition from saying that all the statements are true to saying that $\forall x \alpha(x)$ is true is trivial" [8, p. 192]. The second circumstance concerns the fact that "the argument for the truth of [$\mathcal{G}$] proceeds under the hypothesis that the formal system in question is consistent" [8, p. 192]. Indeed, the $\omega_\downarrow$-rule allows us to cut the Gordian knot of the consistency hypothesis, for $Cons_{\mathsf{PA}}$ turns out to be (deflationarily) *provable* within $\mathsf{PA}^{\omega_\downarrow}$. In this way, we can get rid of the most cumbersome inflationary commitment, namely the resort to the soundness of PA.

4. Concluding remarks

The problem addressed in this paper is that of showing that a deflationary view of incompleteness (incompletability, indeed) is possible, so that the Gödel phenomena are not disastrous for deflationism about truth. We do not mean to argue in favour of a reappraisal of Hilbert's foundational program through the constructive ω-logic as for instance Detlefsen does [7]. What we claim is that the introduction of the constructive ω-rule for achieving $\mathcal{G}$ is in tune with a deflationary point of view: the path leading to $\mathcal{G}$ consists of a proof of $\mathcal{G}$ which drops any reference to a genuine property of 'truth'. The fact that the proofs of $\mathcal{G}$ and $Cons_{\mathsf{PA}}$ within $\mathsf{PA}^{\omega_\downarrow}$ are not relevant to a foundational point of view needs not worry us, as these proofs exploit the First and Second Incompleteness Theorems which *assume* the consistency of PA. But such a foundational irrelevance pinpoints the deflationary nature of the present proposal, by spelling out the irrelevance of the truth value of $\mathcal{G}$ to the grasp of incompleteness phenomena.

Moreover, the deflationary character of our proposal can also be stressed from a broader perspective. If we look at the historical development of number theory — and so assuming the point of view of the mathematical practice — the prototypical reasoning turns out to be a very weak arithmetical demonstrative strategy. It seems indeed that only trivial arithmetical statements can be proved through pure prototype arguments[8].

This is the case, to pick one example, of the proof of the transitivity of the divisibility property: *for all* $a, b, c \in \mathbb{N}$*: if* $a|b$ *and* $b|c$*, then* $a|c$. On this view, the weakness of the $\omega_{\downarrow}$-rule has to be read in opposition to the stronger number theoretical methods of induction and infinite descent employed for proving relevant arithmetical properties [33]. This aspect can be fully grasped by observing that both mathematical induction and infinite descent presuppose a prototype argument. Consider for instance the demonstrative method of mathematical induction. As remarked by Longo, an implicit prototypical passage is silently at work when one proves that the inductive step $\alpha(n) \to \alpha(n+1)$ holds for all $n \in \mathbb{N}$ [24]. Similarly, any proof by descent is built over protypical assumptions. On the one hand, this leads to the conclusion that one of the favourite inflationary tenets — the idea that any demonstrative method non-formalisable within PA has to be epistemologically stronger than the methods encompassed by PA — is shown to be flawed. On the other hand, we get a strong epistemological reason for deflationarily accepting the $\omega_{\downarrow}$-rule: its refusal would imply the refusal of the induction principle itself.

In conclusion, what emerges from our proposal is a notion of deflationism which clearly sponges on the epistemological authority of the mathematical practice, specifically that of number theory. Indeed, the deflationary licitness of the constructive ω-rule has been supported by stressing its undeniable *status* of universally accepted method in the practice of number theory: *explicitly* used through pure applications of prototype arguments, *implicitly* at work as an hidden basic subprinciple in proofs by induction or descent. The same epistemological move provides one further reason for discarding the unrestricted ω-rule to the extent that it expresses a purely semantical principle, too much abstractly-shaped for exhibiting any kind of paradigmatic application *in corpore vili*.

Appendix A. Technical backgrounds and proofs

A.1. **Peano Arithmetic: theory and models.**

Definition 1 (Peano Arithmetic)**.** The language of PA is given by the language of first order logic with identity enriched with the individual constant $\overline{0}$, the unary functional symbol $Succ(\_)$ (the successor) and the two binary functional symbols $+$ and $\cdot$. Moreover, the specific deductive apparatus of PA is defined by the following nine proper axioms:

(1) $x = y \to (x = z \to y = z)$
(2) $x = y \to Succ(x) = Succ(y)$
(3) $\overline{0} \neq Succ(x)$
(4) $Succ(x) = Succ(y) \to x = y$
(5) $x + \overline{0} = x$

[8] The situation seems to be radically different in geometry. Indeed, the very recognition that a deduction about *particular* constructions produces knowledge of *general* validity is at the heart of the emergence of Greek deductive mathematics. For example, the proof of the statement that in every triangle the sum of the three angles is equal to 180° considers a generic triangle, while holding uniformly for all triangles. There is considerable plausibility in Reviel Netz's idea that the feeling of generality that Greek mathematicians gain at the end of a proof arises from the conviction that the proof concerned with a particular object is *repeatable* for any similar object [25, p. 256, 269]. This explanation makes Greek proofs prototype proofs *avant la lettre*.

(6) $x + Succ(y) = Succ(x + y)$
(7) $x \cdot \overline{0} = \overline{0}$
(8) $x \cdot Succ(y) = (x \cdot y) + x$
(9) For every formula $\alpha(x)$ of PA such that x occurs free in α,
$\vdash_{\mathsf{PA}} \alpha(\overline{0}) \to (\forall x(\alpha(x) \to \alpha(Succ(x)) \to \forall x \alpha(x))$.

We abridge with $\overline{n}$ the numeral $Succ(Succ \ldots Succ(0) \ldots)$ resulting from n applications of the successor function to the constant $\overline{0}$. For any pair of terms t and s, $t \neq s$ is intended to be defined as $\neg(t = s)$.

Definition 2 (structure $\mathcal{N}$)**.** The structure $\mathcal{N} = (\mathbb{N}, 0, +^{\mathcal{N}}, \cdot^{\mathcal{N}}, Succ^{\mathcal{N}})$ is formed by the set of non-negative integers $\mathbb{N} = \{0, 1, 2, \ldots\}$, the distinguished number $0 \in \mathbb{N}$ (which interprets the constant $\overline{0}$), the functional symbols $+^{\mathcal{N}}$ and $\cdot^{\mathcal{N}}$ respectively corresponding to the familiar sum and product and the successor function $Succ^{\mathcal{N}}(x) \overset{def.}{=} x +^{\mathcal{N}} 1$.

Theorem A.1 (soundness)**.** *$\mathcal{N}$ is a model of PA and, moreover, PA is sound w.r.t. $\mathcal{N}$.*

Proof. For establishing that $\mathcal{N}$ is a model of PA (in symbols $\mathcal{N} \vDash \mathsf{PA}$) we have to show that each of PA axioms is interpreted in $\mathcal{N}$ as a true statement. It is immediate to check that $\mathcal{N}$ satisfies axioms 1–8. As far as the induction principle is concerned (axiom 9), since the domain of $\mathcal{N}$ exactly coincides with the set of naturals $\mathbb{N}$, the inductive mechanism is indeed able to cover the totality of the elements of $\mathcal{N}$, so as to justify the introduction of the universal quantifier.

As far as the soundness property is concerned, the proof consists in showing that, for any formula α: if $\vdash_{\mathsf{PA}} \alpha$, then $\mathcal{N} \vDash \alpha$. We proceed by induction on the length of the PA proof ending with $\vdash_{\mathsf{PA}} \alpha$. The base is clearly provided by the fact that $\mathcal{N} \vDash \mathsf{PA}$. Then, it is easy to see that the logical inference rules transmit the truth from premisses to conclusions. □

Remark 1 (soundness and consistency). Due to the strict bivalence of classical semantics, if a theory is sound w.r.t. a certain model, it is consistent (otherwise the theory would prove a false statement).

Remark 2 (standard model). The structure $\mathcal{N}$ is said to be the *standard model* for PA.

A.2. **The incompleteness theorems.**

Definition 3 (deductive independence)**.** A formula α is said to be independent of PA if $\nvdash_{\mathsf{PA}} \alpha$ and $\nvdash_{\mathsf{PA}} \neg\alpha$.

Definition 4 (ω-consistency)**.** A certain arithmetical theory T is said to be ω-consistent if the following two conditions are mutually excluding:

- for all $n \in \mathbb{N}$, $\vdash_{\mathsf{T}} \alpha(\overline{n})$,
- $\vdash_{\mathsf{T}} \exists x \neg\alpha(x)$.

Remark 3. ω-consistency is stronger than consistency so like any ω-consistent theory is also consistent.

The proofs of the incompleteness theorems are here merely sketched; for the technical details the reader is referred to [26].

Theorem A.2 (First Incompleteness Theorem)**.** *There exists a formula $\mathcal{G}$ such that, if* PA *is ω-consistent, then $\mathcal{G}$ is independent of* PA.

Proof. The proof is developed through the following five points.

(1) There exists a 1-1 assignment of natural numbers to formulas and demonstrations of PA. $\ulcorner\alpha\urcorner$ and $\overline{\ulcorner\alpha\urcorner}$ respectively indicate the number associated with α (its Gödelian code) and its corresponding numeral: if $\ulcorner\alpha\urcorner = n$, then $\overline{\ulcorner\alpha\urcorner} = \overline{n}$. In the same way, $\ulcorner\pi\urcorner$ and $\overline{\ulcorner\pi\urcorner}$ respectively denote the Gödelian code of the proof π and the corresponding numeral.

(2) It is possible to define a Δ_0-formula $Dem_{\mathsf{PA}}(x, y)$ such that $\vdash_{\mathsf{PA}} Dem_{\mathsf{PA}}(\overline{n}, \overline{m})$ if, and only if, n encodes a PA demonstration of the formula α with $\ulcorner\alpha\urcorner = m$.

(3) Consider the predicate $Theor_{\mathsf{PA}}(y) \overset{def.}{=} \exists x Dem_{\mathsf{PA}}(x, y)$. Its negation admits a formula $\mathcal{G}$ as a fixed point, i.e.:

$$\vdash_{\mathsf{PA}} \mathcal{G} \leftrightarrow \neg Theor_{\mathsf{PA}}(\overline{\ulcorner\mathcal{G}\urcorner}).$$

(4) $\vdash_{\mathsf{PA}} \mathcal{G}$ implies $\vdash_{\mathsf{PA}} \neg\mathcal{G}$ and so, if PA is consistent, $\nvdash_{\mathsf{PA}} \mathcal{G}$.

(5) If PA is ω-consistent, then $\nvdash_{\mathsf{PA}} \mathcal{G}$ implies $\nvdash_{\mathsf{PA}} \neg\mathcal{G}$.

Finally, $\mathcal{G}$ is independent of PA. □

Theorem A.3 (Second Incompleteness Theorem)**.** *Consider the formula*

$$Cons_{\mathsf{PA}} \equiv \neg Theor_{\mathsf{PA}}(\ulcorner 0 = 1\urcorner)$$

asserting the consistency of PA*: it is independent from* PA *as well as $\mathcal{G}$.*

Proof. The proof consists in showing that $Cons_{\mathsf{PA}}$ is provably equivalent to $\mathcal{G}$, i.e. $\vdash_{\mathsf{PA}} Cons_{\mathsf{PA}} \leftrightarrow \mathcal{G}$. In such a way, $\vdash_{\mathsf{PA}} Cons_{\mathsf{PA}}$ and $\vdash_{\mathsf{PA}} \neg Cons_{\mathsf{PA}}$ would respectively imply $\vdash_{\mathsf{PA}} \mathcal{G}$ and $\vdash_{\mathsf{PA}} \neg\mathcal{G}$, against the First Incompleteness Theorem. □

A.3. Σ_1-completeness and related results.

Definition 5 (logical complexity)**.**
- A formula α belongs to the set Δ_0 if it is equivalent to a closed formula α' in which all the quantifiers, if any, are bounded.
- A formula α belongs to Σ_1 (resp. Π_1) if it is equivalent to a closed formula $\alpha' \equiv \exists x\beta(x)$ (resp. $\alpha' \equiv \forall x\beta(x)$) such that $\beta[t/x] \in \Delta_0$.
- A formula α belongs to Σ_{n+1} (resp. Π_{n+1}) if it is equivalent to a closed formula $\alpha' \equiv \exists x\beta(x)$ (resp. $\alpha' \equiv \forall x\beta(x)$) such that $\beta[t/x] \in \Pi_n$ (resp. $\beta[t/x] \in \Sigma_n$).

Example A.1. Both the Gödelian propositions $\mathcal{G}$ and $Cons_{\mathsf{PA}}$ are Π_1-statement.

Remark 4. Whereas $\alpha \in \Sigma_n$ if, and only if, $\neg\alpha \in \Pi_n$, the set of Δ_0 formulas is closed under negation.

Proposition A.4. *Let t, s be two closed arithmetical terms:*

(1) if $\mathcal{N} \vDash t = s$, then $\vdash_{\mathsf{PA}} t = s$,
(2) if $\mathcal{N} \vDash t \neq s$, then $\vdash_{\mathsf{PA}} t \neq s$,
(3) $\vdash_{\mathsf{PA}} \overline{n} \geqslant \overline{m} \to (\overline{n} = \overline{0} \vee \overline{n} = \overline{1} \vee \ldots \vee \overline{n} = \overline{m})$.

Proof. The reader can find all the proofs in [26]. □

Theorem A.5 (Δ_0-decidability)**.** *If α is a closed Δ_0-formula, then either $\vdash_{\mathsf{PA}} \alpha$ or $\vdash_{\mathsf{PA}} \neg\alpha$.*

Proof. Let $\alpha \in \Delta_0$; we proceed by induction on the number of logical connectives occurring in α.

Base. If no logical connective occurs in α, then $\alpha \equiv t = s$ with t, s closed terms. It is either $\mathcal{N} \vDash t = s$ or $\mathcal{N} \vDash t \neq s$ and so Proposition A.4 gives us the basis.

Step. Proposition A.4 enables us to stress the following conversions

$\exists x \leqslant k\alpha(x) \Leftrightarrow \alpha(0) \vee \alpha(1) \vee \ldots \vee \alpha(k)$

$\forall x \leqslant k\alpha(x) \Leftrightarrow \alpha(0) \wedge \alpha(1) \wedge \ldots \wedge \alpha(k)$,

for turning any quantified Δ_0-formula into an equivalent one without quantifiers. Then it is easy to see that any Boolean composition of decidable propositions is, in turn, decidable. □

Corollary A.6 (Δ_0-completeness)**.** *For any closed $\alpha \in \Delta_0$, if $\mathcal{N} \vDash \alpha$, then $\vdash_{\mathsf{PA}} \alpha$.*

Proof. Let $\mathcal{N} \vDash \alpha$, but $\nvdash_{\mathsf{PA}} \alpha$. For $\alpha \in \Delta_0$, by Theorem A.5, it would be $\vdash_{\mathsf{PA}} \neg\alpha$ against the soundness of PA w.r.t. $\mathcal{N}$. □

Theorem A.7. *PA is Σ_1-complete w.r.t. $\mathcal{N}$ if, and only if, it is Δ_0-decidable.*

Proof. ($\Rightarrow$) Let $\nvdash_{\mathsf{PA}} \alpha$, with α closed and in Δ_0. By the Σ_1-completeness, we obtain $\mathcal{N} \nvDash \alpha$ and so $\mathcal{N} \vDash \neg\alpha$. Since $\neg\alpha \in \Delta_0$, we perform a further step of Σ_1-completeness so as to obtain $\vdash_{\mathsf{PA}} \alpha$.

($\Leftarrow$) We proceed by absurd: let $\exists x\alpha(x)$ be a closed Σ_1-formula such that $\mathcal{N} \vDash \exists x\alpha(x)$, but $\nvdash_{\mathsf{PA}} \exists x\alpha(x)$. For $\mathcal{N} \vDash \exists x\alpha(x)$, there is an $n \in \mathbb{N}$ such that $\mathcal{N} \vDash \alpha(n)$. Since $\alpha(n) \in \Delta_0$, we can apply the just proved Δ_0-completeness and obtain $\vdash_{\mathsf{PA}} \alpha(\overline{n})$. As a matter of logic, we finally obtain $\vdash_{\mathsf{PA}} \exists x\alpha(x)$ which contradicts our assumption that $\nvdash_{\mathsf{PA}} \exists x\alpha(x)$. □

Corollary A.8 (Σ_1-completeness)**.** *PA is Σ_1-complete w.r.t. $\mathcal{N}$.*

Proof. Straightforwardly by Theorems A.5 and A.7. □

Corollary A.9. *If $\alpha \in \Pi_1$ is independent of PA, then $\mathcal{N} \vDash \alpha$. In particular, we have that $\mathcal{N} \vDash \mathcal{G}$ and $\mathcal{N} \vDash Cons_{\mathsf{PA}}$.*

Proof. By the Σ_1-completeness, we obtain $\mathcal{N} \nvDash \neg\alpha$ from $\nvdash_{\mathsf{PA}} \neg\alpha$, and so $\mathcal{N} \vDash \alpha$. Both the Gödelian propositions $\mathcal{G}$ and $Cons_{\mathsf{PA}}$ instantiate the case just explained so that $\mathcal{N} \vDash \mathcal{G}$ and $\mathcal{N} \vDash Cons_{\mathsf{PA}}$. □

A.4. ω-logic, constructive ω-logic and some related results.

Theorem A.10. *For any formula α: $\mathcal{N} \vDash \alpha$ if, and only if, $\vdash_{\mathsf{PA}^\omega} \alpha$.*

Proof. (Soundness) It is a matter of extending the proof of Theorem A.1 so as to include the ω-rule. In order to show that any instance of the ω-rule transmits the truth from premisses to the conclusion it is sufficient to remark that the ω-rule just provides a syntactical rendition of the Tarskian definition of the universal quantifier: if $\mathcal{N} \vDash \alpha(0)$, $\mathcal{N} \vDash \alpha(1)$, $\mathcal{N} \vDash \alpha(2)$ and so on, then $\mathcal{N} \vDash \forall x\alpha(x)$.

(Completeness) We proceed by induction on the logical complexity of α. The Δ_0-completeness provides the base of our induction. Then, we distinguish two cases.

- Let $\alpha \equiv \exists x \beta(x) \in \Sigma_{n+1}$. $\mathcal{N} \vDash \exists x \beta(x)$ means that there is an $n \in \mathbb{N}$ such that $\mathcal{N} \vDash \beta(n)$ with $\beta(n) \in \Pi_n$. By inductive hypothesis $\vdash_{\mathsf{PA}^\omega} \beta(\overline{n})$ and so we can introduce the existential quantifier for finally achieving $\vdash_{\mathsf{PA}^\omega} \exists x \beta(x)$.
- Let $\alpha \equiv \forall x \beta(x) \in \Pi_{n+1}$. $\mathcal{N} \vDash \forall x \beta(x)$ means that, for all $n \in \mathbb{N}$, $\mathcal{N} \vDash \beta(n)$ with $\beta(n) \in \Sigma_n$. By inductive hypothesis we have that, for all $n \in \mathbb{N}$, $\vdash_{\mathsf{PA}^\omega} \beta(\overline{n})$. Finally, the ω-rule enables us to introduce the universal quantifier so as to obtain $\vdash_{\mathsf{PA}^\omega} \forall x \beta(x)$.

□

Corollary A.11. PA^ω *is syntactically complete, namely, for any formula* α*, either* $\vdash_{\mathsf{PA}^\omega} \alpha$ *or* $\vdash_{\mathsf{PA}^\omega} \neg\alpha$*.*

Proof. We show that $\nvdash_{\mathsf{PA}^\omega} \alpha$ implies $\vdash_{\mathsf{PA}^\omega} \neg\alpha$. Let $\nvdash_{\mathsf{PA}^\omega} \alpha$, by Theorem A.10 it is $\mathcal{N} \nvDash \alpha$ and so $\mathcal{N} \vDash \neg\alpha$. Then another application of Theorem A.10 allows us to conclude that $\vdash_{\mathsf{PA}^\omega} \neg\alpha$. □

Theorem A.12. *For any formula* α*: if* $\nvdash_{\mathsf{PA}^{\omega\downarrow}} \alpha$*, then* $\vdash_{\mathsf{PA}^{\omega\downarrow}} \neg Theor_{\mathsf{PA}}(\overline{\ulcorner\alpha\urcorner})$*.*

Proof. Suppose by absurd that there is an $n \in \mathbb{N}$ such that $\vdash_{\mathsf{PA}} Dem_{\mathsf{PA}}(\overline{n}, \overline{\ulcorner\alpha\urcorner})$. This latter would imply the existence of a PA proof π of α such that $\ulcorner\alpha\urcorner = n$. This is in contrast with our hypothesis that $\nvdash_{\mathsf{PA}^{\omega\downarrow}} \alpha$ and so we conclude $\nvdash_{\mathsf{PA}} Dem_{\mathsf{PA}}(\overline{n}, \overline{\ulcorner\alpha\urcorner})$. Then, the Δ_0–decidability allows to turn $\nvdash_{\mathsf{PA}} Dem_{\mathsf{PA}}(\overline{n}, \overline{\ulcorner\alpha\urcorner})$ into $\vdash_{\mathsf{PA}} \neg Dem_{\mathsf{PA}}(\overline{n}, \overline{\ulcorner\alpha\urcorner})$. The argument just explained is clearly prototypical w.r.t. n (being, in turn, the proof of Theorem A.5 prototypical w.r.t. the formula α) so as, by a step of $\omega_\downarrow$-rule, we can conclude $\vdash_{\mathsf{PA}^{\omega\downarrow}} \forall x \neg Dem_{\mathsf{PA}}(x, \overline{\ulcorner\alpha\urcorner})$, that is $\vdash_{\mathsf{PA}^{\omega\downarrow}} \neg Theor_{\mathsf{PA}}(\overline{\ulcorner\alpha\urcorner})$. □

Corollary A.13. $\mathsf{PA}^{\omega\downarrow}$ *decides both the Gödelian propositions* $\mathcal{G}$ *and* $Cons_{\mathsf{PA}}$*.*

Proof. Suppose by absurd that $\mathcal{G}$ is not provable in $\mathsf{PA}^{\omega\downarrow}$. By Theorem A.12, we would obtain $\vdash_{\mathsf{PA}^{\omega\downarrow}} \neg Theor_{\mathsf{PA}}(\overline{\ulcorner\mathcal{G}\urcorner})$ from $\nvdash_{\mathsf{PA}^{\omega\downarrow}} \mathcal{G}$. Now, we know that $\vdash_{\mathsf{PA}^{\omega\downarrow}} \mathcal{G} \leftrightarrow \neg Theor_{\mathsf{PA}}(\overline{\ulcorner\mathcal{G}\urcorner})$ and so we would be able to deduce $\vdash_{\mathsf{PA}^{\omega\downarrow}} \mathcal{G}$ against the fact that we assumed $\nvdash_{\mathsf{PA}^{\omega\downarrow}} \mathcal{G}$. Such an argument leads us to reject $\nvdash_{\mathsf{PA}^{\omega\downarrow}} \mathcal{G}$, that is to affirm $\vdash_{\mathsf{PA}^{\omega\downarrow}} \mathcal{G}$.

The proof of $\vdash_{\mathsf{PA}^{\omega\downarrow}} Cons_{\mathsf{PA}}$ proceeds in an analogous way. □

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