How much is worth to remember? A taxonomy based on Petri Nets Unfoldings*

G. Michele Pinna

Dipartimento di Matematica e Informatica
Università degli Studi di Cagliari, Cagliari, Italy
gmpinna@unica.it

Abstract. The notion of unfolding plays a major role in the so called *non sequential* semantics of Petri nets, as well as in model checking of concurrent and distributed systems or in control theory. In literature various approaches to this notion have been proposed, where dependencies among events are represented either taking into account the whole history of the event (the so called *individual token philosophy*) or considering the whole history irrelevant (the so called *collective token philosophy*). In this paper we propose two unfoldings where the history is partially kept. These notions are based on *unravelling* a net rather than *unfolding* it. We compare them with the classical ones and we put all of them together in a coherent framework.

1 Introduction

Petri nets are since almost 40 years one among the most widely used models for concurrent and distributed computations.

This kind of computations can be described at different granularities, remembering to some extent *how* events happen, which in the case of Petri nets usually means either considering the tokens consumed and produced by happening of transitions as *entities* coding the history (namely the occurrences of transitions and the ordering of these occurrences, that have *contributed* to produce the specific token) or simply as *anonymous* ones (the history is forgotten). Henceforth to describe (and to reason on) computations in this setting two main approaches are usually taken: one asserts that each happening of a transition depends on how tokens are consumed and produced by other transitions, and a second declares that the happening of a transitions depends only on the fulfillment of certain conditions, namely the presence of tokens in the preset of the transition, without any concern on how these conditions were produced. In other words, one approach considers the history relevant whereas the other considers the history irrelevant.

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To make more precise this concept, consider the net in Fig. 1. In the net N the transition a depends on the happening of b or c, and if the history would be considered relevant, the information on which of b or c is actually executed must be kept, whereas if the history here does not play a major role then the information on *which of the two transitions* was executed is negligible. The history respecting approach is mainly represented by the classical notion of unfolding (which will be called I-unfolding) as introduced by Winskel ([1])[1]. In this case the transition a depending on b is *different* from the transition a depending on c. This view is reflected naturally in the again classical notion of *prime event structure* ([1]). Concerning the other approach, van Glabbeek and Plotkin in [5] introduced the notion of 1-unfolding (which we call C-unfolding) to model the irrelevance of the history and contextually introduced a notion of event structures, called *configuration structures*, where this principle is reflected.

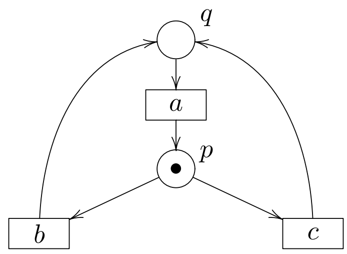

Fig. 1. The net N

In this paper the focus is on the possibility that tokens code just a *part* of the history, hence on notions of unfolding where only parts of the history are kept.

Other authors have addressed this issue, mainly inspired by Petri Nets but focussing on other models: Gunawardena in [6] introduced a notion, called *Muller unfolding* (for the net N this notion is based on the intuition that the $i - th$ happening of a depends on the happening of $i - 1$ occurrences of b or c)[2], and introduced a suitable event based semantics, expressing a principle of *history irrelevance*. In fact already in [7] and [8] Gunawardena advocates the principle of irrelevance of the whole history by introducing *causal automata*, where the so called *or-causality* can be easily modeled, meaning that an event may have several *alternative* causes. In [9] a notion of event structure is introduced, where the dependencies among events are modeled in a local fashion, implementing again a limited principle of history irrelevance, whereas in [10] a notion of scenario is introduced to adjust some weaknesses of classical partial order semantics for Petri nets, where again the history is not posed as the central notion.

Turning to behaviours described in term of suitable nets, Khomenko, Kondratyev, Koutny and Vogler in [11] have developed a way to obtain a more compact representation of the net behaviours, basically identifying all the conditions representing the *same* token and identifying as well the transitions that consume and produce the same conditions (i.e., tokens). Their target was mainly to achieve representations which are more compact, and forgetting part of the

[1] Also other authors have investigated on this notion, e.g. Engelfriet [2] for the unfolding of safe nets, or Baldan, Corradini and Montanari in [3] for the unfolding of net with read arcs, or Baldan, Busi, Corradini and the author in [4] for unfoldings of safe nets with inhibitor and read arcs.

[2] For Gunawardena the word *unfolding* does not refer to a net.

history leads to this kind of results. Fabre in [12] introduces the notion of *Trellis processes*, which is basically an unfolding where conflicting occurrences of a transition are identified, where conflicting means belonging to different computations. Trellis processes are defined for a suitable class of safe nets, namely nets that can be obtained by the union of state-machines, i.e., condition/events nets. Though some similarities with our approach exists, in Trellis a transition codes histories which can be considered as adhering to the individual token philosophy.

In [13] van Glabbeek analyzes the computational interpretations of various firing rules for Petri Nets, distinguishing not only between the individual token approach (the one where history is relevant) and the collective token one (where history is not relevant), but also the fact that a transition may be concurrent with itself. Here we rule out auto-concurrency and then we focus on *safe* nets, i.e., nets where each place in each reachable marking contains at most one token.

The notions of unfoldings introduced in literature are based on suitable nets enjoying *structural* and *behavioural* restrictions: the individual unfoldings on *causal* net, i.e., acyclic nets which can be equipped with an irreflexive conflict relation, the collective unfolding on the notion of *occurrence* net (which can be cyclic), where it is guaranteed that each transition fires just once. The notions of unfolding we will introduce are based on *unravel* nets, namely occurrence nets which are acyclic only when considering suitable subsets of events (representing a computation). Concerning the two new notions of unfoldings, one is based on the observation that the i-th occurrences of a token in a place can be equated (thus in this unfolding, which we will call R_i-unfolding, relative to the net in Fig. 1 the token produced in the place q by either c or b is represented by a unique condition); whereas the other considers also that the all the events happening at the same *depth* (a distance from the initial marking) and having the same *surrounding* can be identified (thus in this unfolding, which we will call R_c-unfolding, relative to the net in Fig. 1 the two occurrence of the transition a, one depending on b and the other on c are identified).

These notions could be obtained by first constructing the individual unfolding, and then identifying suitable places and transitions. We do take another way, as we define these unfolding from scratch, without constructing first the individual unfolding, which may be a complex operation. When identifying the events of both unfoldings we face the problem of determining the *amount* of history that should be recorded: in the R_i-unfolding an event basically codes the count of the transitions in its neighborhood, whereas in the R_c-unfolding it is enough to count just how many times the transition corresponding to the event has been executed.

Beside comparing the two notion of unfoldings introduced in literature, similarly to what van Glabbeek has done in [13] with respect to firing sequences, we discuss also the relative expressiveness of the two new notions. As evidence of the expected relative expressiveness, we will show that what we will call the *individual unfolding*, i.e. the unfolding for the individual token philosophy can be folded onto the other notions of unfoldings, leading to the *collective unfolding* (i.e. the unfolding for the collective token philosophy). The results can be summarized as follow: the I-unfolding can be folded onto the R_i-one, which in turn is folded onto the R_c-unfolding and finally onto the C-unfolding. This amount to saying that histories can be progressively *faded*, i.e., forgotten. In [14] we have shown also that the collective unfolding can be unfolded onto the individual one. And this account to say that the history can be always recovered.

The paper is organized as follows: in the next section we recall the notions we will use in the rest of the paper, and introduce new ones, like the one of unravel net. In section 3 we introduce the notions of unfolding which we will compare in section 4. We sum up our finding in the last section.

2 Nets

Notations: With $\mathbb{N}$ we denote the set of natural numbers and with $\mathbb{N}^+$ the set of natural numbers without zero, i.e., $\mathbb{N} \setminus \{0\}$. Let X be a set and $n \in \mathbb{N}^+$, with X^n we denote the set of n-tuple of elements in X. Given a tuple $\alpha \in X^n$, with $\alpha(i)$ we indicate the i-th entry of the tuple. Let A be a set, a *multiset* of A is a function $m : A \to \mathbb{N}$. The set of multisets of A is denoted by μA. The usual operations and relations on multisets, like multiset union $+$ or multiset difference $-$, are used. We write $m \leq m'$ if $m(a) \leq m'(a)$ for all $a \in A$. If $m \in \mu A$, we denote by $[\![m]\!]$ the multiset defined as $[\![m]\!](a) = 1$ if $m(a) > 0$ and $[\![m]\!](a) = 0$ otherwise; sometimes $[\![m]\!]$ will be confused with the corresponding subset $\{a \in A \mid [\![m]\!](a) = 1\}$ of A. A *multirelation* f from A to B (often indicated as $f : A \to B$) is a multiset of $A \times B$. We will limit our attention to finitary multirelations, namely multirelations f such that the set $\{b \in B \mid f(a,b) > 0\}$ is finite. Multirelation f induces in an obvious way a function $\mu f : \mu A \to \mu B$, defined as $\mu f(\sum_{a \in A} n_a \cdot a) = \sum_{b \in B} \sum_{a \in A} (n_a \cdot f(a,b)) \cdot b$ (possibly partial, since infinite coefficients are disallowed). If f satisfies $f(a,b) \leq 1$ for all $a \in A$ and $b \in B$, i.e. $f = [\![f]\!]$, then we sometimes confuse it with the corresponding set-relation and write $f(a,b)$ for $f(a,b) = 1$.

Given an alphabet Σ, with Σ^* we denote as usual the set of words on Σ, and with $\preceq$ we denote the lexicographic order on it. The length of a word is defined as usual and it is denoted with $[\cdot]$. The number of occurrences of a symbol $a \in \Sigma$ in a word w is standardly defined as $[w]_a = 0$ if $w = \epsilon$, $[w]_a = [x]_a + 1$ if $w = ax$, $[w]_a = [x]_a$ otherwise. Given a word w and a subset A of the alphabet, $\|w\|_A$ is a word obtained deleting all occurrences of symbols not belonging to A, i.e. $\|w\|_A = a\|x\|_A$ if $a \in A$ and $w = ax$, and $\|w\|_A = \|x\|_A$ otherwise. Finally, given a word $w \in \Sigma^+$, with *last*(w) we denote the last symbol appearing in w.

Given a partial order $(D, \sqsubseteq)$, with $\lfloor d \rfloor$ we denote the set $\{d' \in D \mid d' \sqsubseteq d\}$.

Nets and morphisms: We first review the notions of Petri net and of the token game. Then we recall the notion of morphism. Nets and morphism are a category, but our focus is not on the categorical approach to nets, but rather on the various relations among the notions of unfolding that are definable on nets. In this perspective morphisms are quite handy in highlighting how nets (unfoldings) are related.

Definition 1. *A* Petri net *is a 4-tuple* $N = \langle S, T, F, m \rangle$*, where*

- S *is a set of* places *and* T *is a set of* transitions *(with* $S \cap T = \emptyset$*),*
- $F = \langle F_{pre}, F_{post} \rangle$ *is a pair of multirelations from* T *to* S*, and*
- $m \in \mu S$ *is called the* initial marking*.*

We require that for each $t \in T$, $F_{pre}(t, s) > 0$ for some place $s \in S$. Subscripts on the net name carry over the names of the net components. As usual, given a finite multiset of transitions $A \in \mu T$ we write ${}^\bullet A$ for its *pre-set* $\mu F_{pre}(A)$ and $A^\bullet$ for its *post-set* $\mu F_{post}(A)$. The same notation is used to denote the functions from S to $\mathbf{2}^T$ defined as ${}^\bullet s = \{t \in T \mid F_{post}(t, s) > 0\}$ and $s^\bullet = \{t \in T \mid F_{pre}(t, s) > 0\}$, for $s \in S$.

Let N be a net. A finite multiset of transitions A is enabled at a marking m, if m contains the pre-set of A. Formally, a finite multiset $A \in \mu T$ is *enabled* at m if ${}^\bullet A \leq m$. In this case, to indicate that the execution of A in m produces the new marking $m' = m - {}^\bullet A + A^\bullet$ we write $m\,[A\rangle\,m'$. Steps and firing sequences, as well as reachable markings, are defined in the usual way. The set of reachable markings of a net N is denoted with $\mathcal{M}_N$. Each reachable marking can be obviously reached with a firing sequence where just a transition is executed at each step. Thus also a trace can be associated to it, which is the word on T^* obtained by the firing sequence considering just the transitions and forgetting about the markings. The traces of a net N are denoted with $\mathit{Traces}(N)$.

We recall now the notion of morphism between nets, which we will use to construct an unfolding (and also to relate the various notions of unfolding).

Definition 2. *Let* $N_0 = \langle S_0, T_0, F_0, m_0 \rangle$ *and* $N_1 = \langle S_1, T_1, F_1, m_1 \rangle$ *be nets. A* morphism $h : N_0 \to N_1$ *is a pair* $h = \langle \eta, \beta \rangle$*, where* $\eta : T_0 \to T_1$ *is a partial function and* $\beta : S_0 \to S_1$ *is a multirelation such that*

- $\mu\beta(m_0) = m_1$*,*
- *for each* $t \in T$*,* $\mu\beta({}^\bullet t) = {}^\bullet\eta(t)$*, and* $\mu\beta(t^\bullet) = \eta(t)^\bullet$*.*

The conditions of the above definition are the defining conditions of Winskel's morphisms on ordinary nets ([1]). A morphism $h : N_0 \to N_1$ in this setting is a *simulation*, in the sense that each reachable marking in N_0 is also a reachable marking in N_1.

Proposition 1. *Let* N_0 *and* N_1 *be nets, and let* $h = \langle \eta, \beta \rangle : N_0 \to N_1$ *be a net morphism. For each* $m, m' \in \mathcal{M}_{N_0}$ *and* $A \in \mu T$*, if*

$$m\,[A\rangle\,m' \text{ then } \mu\beta(m)\,[\mu\eta(A)\rangle\,\mu\beta(m').$$

Therefore net morphisms preserve reachable markings, i.e. if m *is a reachable marking in* N_0 *then* $\mu\beta(m)$ *is reachable in* N_1*.*

Safe nets: In this paper we consider only safe nets:

Definition 3. *A net* $N = \langle S, T, F, m \rangle$ *is said* safe *in the case that* F_{pre} *and* F_{post} *are such that* $F_{pre} = [\![F_{pre}]\!]$*,* $F_{post} = [\![F_{post}]\!]$ *and each marking* $m \in \mathcal{M}_N$ *is such that* $m = [\![m]\!]$*.*

Furthermore we require that the safe nets are such that $\forall s \in S, \forall t \in T$ it holds that $F_{pre}(t, s) \neq F_{post}(t, s)$.

Subnet: A subnet of a net is a net obtained restricting places and transitions, and correspondingly also the multirelation F and the initial marking.

Definition 4. *Let $N = \langle S, T, F, m\rangle$ be a Petri net and let $T' \subseteq T$. Then the subnet generated by T' is the net $N|_{T'} = \langle S', T', F', m', \rangle$, where*

- *$S' = \{s \in S \mid F_{pre}(t, s) > 0$ or $F_{post}(t, s) > 0$ for $t \in T'\} \cup \{s \in S \mid m(s) > 0\}$,*
- *$F' = \langle F'_{pre}, F'_{post}\rangle$ is the pair of multirelations from T' to S' obtained restricting $\langle F_{pre}, F_{post}\rangle$ to S' and T',*
- *m' is the multiset on S' obtained by m restricting to places in S'.*

It is trivial to observe that, given a net $N = \langle S, T, F, m\rangle$ and a subnet $N|_{T'}$, with $T' \subseteq T$, then the pair $\eta(t) = t$ and $\beta : S \to S'$ is $\beta(s, s) = 1$ if $s \in S'$ and 0 otherwise is a well defined morphism from N to $N|_{T'}$.

Occurrence Nets: The notion of occurrence net we introduce here is the one called 1-occurrence net and proposed by van Glabbeek and Plotkin in [5]. First we need to introduce the notion of *state*.

Definition 5. *Let $N = \langle S, T, F, m\rangle$ be a Petri net, the* configuration *is any finite multiset X of transitions with the property that the function $m_X : S \to \mathbb{Z}$ given by $m_X(s) = m(s) + \sum_{t\in T} X(t) \cdot (F_{post}(t, s) - F_{pre}(t, s))$, for all $s \in S$, is a marking of the net.*

Given two configurations X, Y, we stipulate that $X \xrightarrow{A} Y$ iff $m_X\,[A\rangle\, m_Y$. A configuration X is reachable iff $X = \oplus_{i=1}^{n} X_i$ is such that $\oplus_{j=1}^{k-1} X_j \xrightarrow{X_k} \oplus_{j=1}^{k} X_j$ for all $1 \le k \le n$. If X is a reachable configuration, we call it a state. *With $\mathcal{X}(N)$ we denote the states of a net.*

A state contains (in no order) all the occurrence of the transitions that have been fired to reach a marking. Observe that a trace of a net is a suitable linearization of the elements of a state X. On the notion of state the notion of occurrence net is based:

Definition 6. *An* occurrence net *$O = \langle S, T, F, m\rangle$ is a Petri net where each state is a set, i.e. $\forall X \in \mathcal{X}(N)$ it holds that $X = [\![X]\!]$.*

The intuition behind this notion is the following: regardless how tokens are produced or consumed, an occurrence net *guarantees* that each transition can *occur* only once (hence the reason for calling them occurrence nets). As suggested in the introduction, the history of a token (how it is produced) is completely forgotten. Observe that, differently from the notion of causal net we will introduce later, an occurrence net is not required in general to be a safe net. Another relevant characteristic is that it is not possible (or at least easy) to obtain directly relationships among occurrences of transitions, as it will be with causal nets.

Causal Nets: The notion of causal net we use here is the classical one, though it is often called occurrence net. The different name is due to the other notion of occurrence net we have introduced above.

Causal nets are structurally safe nets, hence the multirelations F_{pre} and F_{post} can be considered as a *flow* relation $F \subseteq (S \times T) \cup (T \times S)$ by stating $s\ F\ t$ iff $F_{pre}(t, s)$ and $t\ F\ s$ iff $F_{post}(t, s)$. Hence we can use the usual notation for the transitive (and reflexive) closure of this relation. For denoting places and transitions we use B and E (see [15] and [1, 16]). A causal net is essentially an acyclic net equipped with a conflict relation (which is deduced using the transitive closure of F).

Definition 7. *A* causal net $C = \langle B, E, F, m\rangle$ *is a safe net satisfying the following restictions:*

- $\forall b \in m,\ {}^\bullet b = \emptyset$,
- $\forall b \in B.\ \exists b' \in m$ *such that* $b' F^* b$,
- $\forall b \in B.\ |{}^\bullet b| \leq 1$,
- F^+ *is irreflexive and, for all* $e \in E$, *the set* $\{e' \mid e'\ F^* e\}$ *is finite, and*
- $\#$ *is irreflexive, where* $e \#_i e'$ *iff* $e, e' \in E$, $e \neq e'$ *and* ${}^\bullet e \cap {}^\bullet e' \neq \emptyset$, *and* $x \# x'$ *iff* $\exists y, y' \in B \cup E$ *such that* $y \#_i y'$ *and* $y F^* x$ *and* $y' F^* x'$.

The intuition behind this notion is the following: each place b represents the occurrence of a token, which is produced by the *unique* transition in ${}^\bullet b$, unless b belongs to the initial marking, and it is used by only one transition (hence if $e, e' \in b^\bullet$, then $e \# e'$). On causal nets it is easy to define a relation expressing *concurrency*: two elements of the causal net are concurrent if they are neither causally dependent nor in conflict. Formally $x\ co\ y$ iff $\neg(x \# y$ or $x F^+ y$ or $y F^+ x)$. This relation can be extended to sets of conditions: let $A \subseteq B$, then $\mathbf{co}(A)$ iff $\forall b, b' \in A.\ b\ co\ b'$ and $\{e \in E \mid \exists b \in A.\ e F^* b\}$ is finite.

The following observation essentially says that each transition (event) in a causal net is executed only once:

Proposition 2. *let* $C = \langle B, E, F, m\rangle$ *be a causal net, then* C *is also an occurrence net.*

Unravel Nets: Causal nets structurally capture dependencies (and conflict) whereas occurrence nets structurally capture the unique occurrence property of each transition. We introduce now a notion of net which will turn to be, so to say, in between occurrence nets and causal nets. With respect to the notion of occurrence net we want still to structurally assure that each transition happens just once, whereas with respect to causal net we want still to be able to retrieve dependencies among the firings of transitions.

Definition 8. *An* unravel net $R = (\langle S, T, F, m\rangle, \mathcal{P})$ *is a pair where:*

- $\langle S, T, F, m\rangle$ *is an occurrence net,*
- *for each state* $X \in \mathcal{X}(\langle S, T, F, m\rangle)$ *such that* $\mathcal{P}(X)$, *the restriction of* $\langle S, T, F, m\rangle$ *to the transitions in* $[\![X]\!]$, *i.e.,* $\langle S, T, F, m\rangle|_{[\![X]\!]}$, *is a causal net.*

Given and unravel net $R = (\langle S, T, F, m\rangle, \mathcal{P})$, with abuse of notation we will often use R for $\langle S, T, F, m\rangle$.

As discussed previously, the intuition behind an unravel net is the similar of the *merged* processes in ([11]) where each place of the net represents a token (as in causal ones) but it can be produced in various ways (the possible histories of the token) and these histories can be recovered once one considers a state of the net. Thus on the one hand, the idea is that the i-th presence of a token in a place is *signaled* just once; and on the other hand the states of the nets satisfying a property (which we left here unspecified) leads to a causal net. Observe that, given an unravel net R and a state X satisfying the required property, $R|_{[\![X]\!]}$ is a causal net where the conflict relation is empty.

This notion covers the one of causal net, as the following proposition shows.

Proposition 3. *Let $C = \langle B, E, F, m\rangle$ be a casual net. Then $R = (C, \mathcal{P}_{\leq})$ is an unravel net, where $\mathcal{P}_{\leq}(X)$ holds iff for all $t \in [\![X]\!]$. $\lfloor x\rfloor \subseteq [\![X]\!]$, with respect to (E, F^*).*

It is worth to observe that the choice of the property turning causal nets into unravel nets is simple. Furthermore other properties on transitions can be used in this case: for instance, the one holding for each subset of transitions, or the one requiring that $[\![X]\!]$ is conflict free, i.e., for all $t, t' \in [\![X]\!]$ it holds that $(t, t') \not\in \#$, where $\#$ is the conflict relation induced by the causal net.

3 Unfoldings

In this section we introduce various notions of unfoldings of a net, related to occurrence, unravel and causal nets. Two of these constructions are *classic* and are closely related to the individual and collective token philosophy, the others are new. In the following we will use the same notation both for the multirelations and for the relations between transitions (events) and places (conditions). This can be done safely as on causal nets these coincide.

Individual unfolding: In the case of the individual token philosophy the unfolding of a net N is a causal net. It can be defined either as top of a chain of causal nets or as the unique causal net satisfying certain conditions.

Definition 9. *Let $N = \langle S, T, F, m\rangle$ be a safe net. The I-*unfolding *$\mathcal{U}_I(N) = \langle B^I, E^I, F^I, m^I\rangle$ is the net defined as follows:*

$$B^I = \{(m, s) \mid s \in S \textit{ and } m(s) > 0\} \bigcup \{(\{e\}, *) \mid e \in E^I\}$$
$$\bigcup \{(\{e\}, s) \mid e \in E^I \textit{ and } s \in S \textit{ and } F^I_{pre}(\eta^I(e), s) > 0\}$$

$$E^I = \{(X, t) \mid X \subseteq B^I \textit{ and } \mathbf{co}(X) \textit{ and } {}^\bullet t = \mu\beta^I(X)\}$$

$$F^I = \begin{cases} F^I_{pre}((X, t), b) \textit{ iff } b \in X \textit{ or } b = (\{(X, t)\}, *) \\ F^I_{post}((X, t), b) \textit{ iff } \exists s \in S, i \in \mathbb{N}^+.\ b = ((X, t), s) \end{cases}$$

$$m^I = \{(m, s) \mid (m, s) \in B^I\} \bigcup \{(\{e\}, *) \mid e \in E^I\}$$

where **co** *is the concurrency relation obtained by* F^I *on* B *and* E*,* $\eta : E^I \to T$ *is defined as* $\eta^I(X, t) = t$ *and* $\beta : B^I \to S$ *is defined as* $\beta^I(X, s) = s$*.*

In this construction each token carries the whole history, and the events as well. As we said before, there are other alternative characterizations of the I-unfolding of a net.

The construction we have introduced has a little difference with respect to the ones introduced in the literature (e.g. [1] or [2]), as we introduce a condition for each event (transition), namely $(\{e\}, *)$, which enforces the unique occurrence property of each event (transition). This cause no harm (these conditions are superfluous, in a net theoretical sense, for causal nets) but they are useful in relating the unfolding constructions.

Proposition 4. *Let* N *be a safe net and* $\mathcal{U}_I(N)$ *its* I-unfolding*. Then* $\mathcal{U}_I(N)$ *is a causal net and* $\langle \eta^I, \beta^I \rangle : \mathcal{U}_I(N) \to N$ *is a well defined net morphism.*

A consequence of this construction is that each reachable marking of the unfolding is a reachable marking of the net (this is due to the folding morphism) but also the converse holds, i.e. to each reachable marking of the net N a reachable marking in the unfolding corresponds.

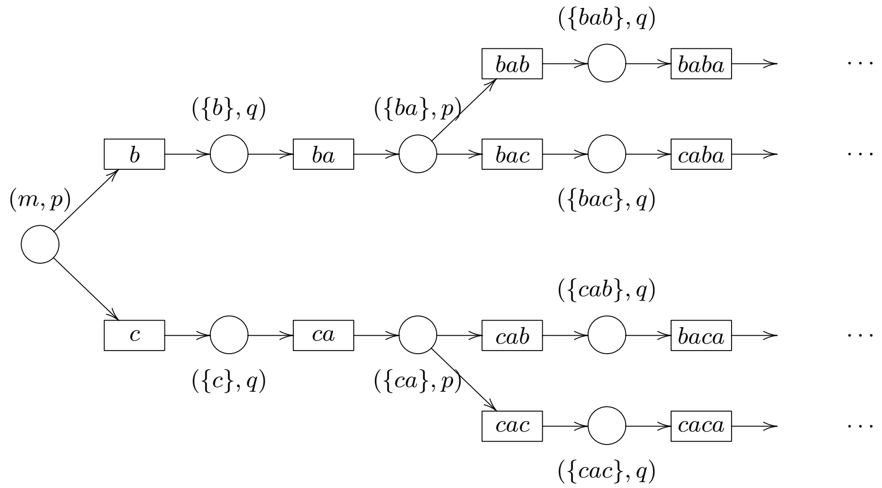

Fig. 2. Part of the I-unfolding of N (places $(t, *)$ are omitted).

R_i-unfolding: We introduce now the first of the two new notions of unfolding of a net, which will turn out to be an unravel net (and we will denote it with $\mathcal{U}_{R_i}$). In order to construct this unfolding, we need some auxiliary notations. Let $N = \langle S, T, F, m \rangle$ be a safe Petri net, and let $t \in T$, with $\circlearrowleft(t) = \{t' \mid t' \in {}^\bullet s$ or

$t' \in s'^{\bullet}$ with $s \in {}^{\bullet}t$ and $s' \in t^{\bullet}\} \cup \{t\}$ we denote the *neighborhood* of t, namely the transitions *following* and *preceding* t, including t.

The places of this unfolding are easy to identify: consider a safe net $N = (S, T, F, m)$, then the places are $\{(s, i) \mid s \in S \text{ and } i \in \mathbb{N}^+\}$, meaning that the place s got its i-th token.

To identify the transitions, let us consider the part of the net depicted in Fig. 3. The i-th firing of the transition a (consuming the j-th token put in s) depends on the number of occurrences of x, y, b and c (and a itself). It clearly depends on x and y as they produce tokens in s, but also on b, c and the previous occurrences of a, as these remove tokens from s. Thus we have to collect in some way this information, which is directly available on the individual unfolding (by looking at the past of the events representing the i-th firing of a). The i-th firing of the transition a (which may have various histories, that for the time being we keep separate) is represented by the event $\{(\!|w|\!)_{\sim}a \mid w \in (\circlearrowleft(a))^*\}$ where $\sim$ is an equivalence relation stating that $w \sim w'$ iff $|w| = |w'|$ and for all $t \in T$ $[w]_t = [w']_t$, and $(\!|w|\!)_{\sim}$ denotes the equivalence class of the word w which is such that

- $[w]_a = i - 1$ (before the i-th happening of a, a itself must have happened $i - 1$-th times),
- $[w]_x + [w]_y$ is equal to j if $m(s) = 0$, or it is equal to $j - 1$ is $m(s) = 1$ (x and y must have happened a number of times (j or $j-1$) in order to put the token the j-th time in s, and
- $[w]_a + [w]_b + [w]_c = j - 1$ (the token has been removed just $j - 1$ times).

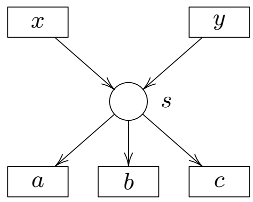

Fig. 3. Part of a safe net

This definition of event takes into account what we can call its *local* history of a, by describing the happening of the transitions in its neighborhood (the order is not relevant and can be retrieved looking among the elements in the equivalence class). We often confuse the equivalence class with one of its elements.

To illustrate how transitions and places are connected, we refer again to the Fig. 3. The transition wa happens whenever the places in its preset are marked, which means that if $s \in {}^{\bullet}a$ in N, then wa can happens when the place (s, j) got marked, with $j = [w]_{{}^{\bullet}s \cup s^{\bullet}} + 1$. Assume that x, y, b and c have happened respectively k, m, n, p times. Then the w is the encoding of this fact (and should have length $k + m + n + p + i - 1$). Thus an arc will connect (s, j) to wa in the unravel net. To characterize which event put the j-th token in s, we have to concentrate our attention on x and y. This token in s is put either by x or y, and precisely by possibly any of the occurrence of x and y, wx or $w'y$. where $j = [wx]_{\{x,y\}}$ and $j = [w'y]_{\{x,y\}}$. Thus an arc should connect these events to (s, j).

What remain to describe is the property $\mathcal{P}$. An unravel net can have cycles (as it should be only *locally* acyclic). Notwithstanding, it is easy to require that the subnet associated to a state X is acyclic, just by requiring that the relation F^* restricted to the elements of X is a partial order. Unfortunately this is not enough. Consider the state of the net in fig. 4 $\{c, ba, bac\}$. The subnet identified by these three events is acyclic, but it is not what we would like to have, at least at this point, as c corresponds in the original net to c and ba conveys the idea that it represents the happening of a after b has happened (and not c as in this case). Thus we require that also the events have to be related, requiring that the events in X can be ordered in a suitable way with respect to their local histories. Given a trace w and an event $(\!|w'|\!)_\sim b$, we can *add* the b at the end of the trace whenever $\|w\|_{\circlearrowleft(b)} \in (\!|w'|\!)_\sim$, and we denote it with $w \circledcirc b$. Let Y be a finite set of events equipped with a total ordering $\sqsubseteq$, with $\Downarrow_\sqsubseteq Y$ we denote the word $(\Downarrow_\sqsubseteq (Y \setminus \{max(Y)\})) \circledcirc last(max(Y))$. Putting all together, the property we are looking for on a state X, is the following:

- X is acyclic with respect to $F^* \cap [\![X]\!] \times [\![X]\!]$, and
- there exists a relation $\lhd$ on $[\![X]\!]$ such that
  - $\lhd^*$ is a total order compatible with F^*, and
  - $\Downarrow_{\lhd^*} [\![X]\!] \in Traces(N)$.

We are now ready to propose the following construction:

Definition 10. *Let $N = \langle S, T, F, m\rangle$ be a safe net. The R_i-unfolding $\mathcal{U}_{R_i}(N) = (\langle S^{R_i}, T^{R_i}, F^{R_i}, m^{R_i}\rangle, \mathcal{P}^{R_i})$ is the net defined as follows:*

$$S^{R_i} = (S \times \mathbb{N}^+) \cup (T^{R_i} \times \{*\})$$

$$T^{R_i} = \bigcup_{t\in T}\{(\!|w|\!)_\sim t \mid w \in (\circlearrowleft(t))^*\}$$

$$F^{R_i} = \begin{cases} F^{R_i}_{pre}((\!|w|\!)_\sim t, (s,k)) & \textit{iff } F_{pre}(t,s) \\ & \textit{and } k = [\|w\|_{s^\bullet \cup {}^\bullet s}] + m^{R_i}(s,1) \\ F^{R_i}_{pre}((\!|w|\!)_\sim t, ((\!|w|\!)_\sim t, *)) & \\ F^{R_i}_{post}((\!|w|\!)_\sim t), (s,k)) & \textit{iff } F_{post}(t,s) \\ & \textit{and } k = [\|w\|_{{}^\bullet s}] + m^{R_i}(s,1) + 1 \end{cases}$$

$$m^{R_i} = \{((s,1)) \mid m(s) > 0\} \bigcup \{(e,*) \mid e \in T^{R_i}\}$$

$\mathcal{P}^{R_i}(X)$ iff X is a state and there exists a total order $\lhd^*$, compatible with $(F^{R_i})^*$, such that $\Downarrow_{\lhd^*} [\![X]\!] \in Traces(N)$.

Furthermore $\eta^{R_i} : T^{R_i} \to T$ is defined as $\eta((\!|w|\!)_\sim t) = t$ and $\beta^{R_i} : S^{R_i} \to S$ is a multirelation defined as $\beta^{R_i}((s,j)) = s$ iff $s \in S$.

Observe that transitions produced by the *firing of the same transitions* in the I-unfolding of a safe net N are identified in the R_i-unfolding (for instance, the two transitions *baca* and *caba* are now the same in this unfolding), and several conditions of the I-unfolding are mapped to a single one in the R_i-unfolding (i.e., those representing the same occurrence of a token in the place). Opposed

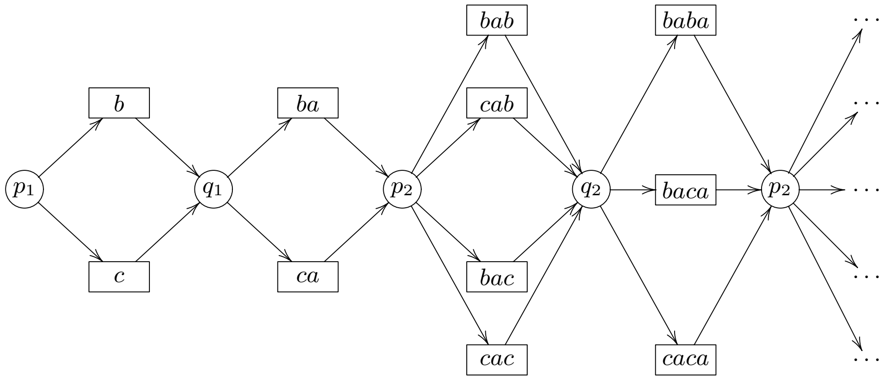

Fig. 4. Part of the R_i-unfolding of N (places $(t, *)$ are omitted).

to the I-unfolding, where each event encodes the whole history, here it encodes just a part of the whole history. The same happening, in different context, of a transition in the original net is represented by an event for each one (see the events ca and ba in the unfolding in Fig. 4. This unfolding has the same conditions of the merged process of a safe net defined in [11], as we will see later (some events that could be identified, are not glued in our approach as for the time being a relevant part of the history is preserved).

The following proposition shows that the construction we have proposed is indeed a faithful representation of the behaviour of a net, as to each reachable marking of the net N a state in the unfolding satisfying the suitable property can be found.

Proposition 5. *Let $N = \langle S, T, F, m\rangle$ be a safe net and $\mathcal{U}_{R_i}(N) = (\langle S^{R_i}, T^{R_i}, F^{R_i}, m^{R_i}\rangle, \mathcal{P}^{R_i})$ its R_i-unfolding. Then*

1. *$\mathcal{U}_{R_i}(N)$ is an unravel net,*
2. *$\langle \eta^{R_i}, \beta^{R_i}\rangle : \mathcal{U}_{R_i}(N) \to N$ is a well defined net morphism, and*
3. *for each state $X \in \mathcal{X}(N)$ there exists a state $Y \in \mathcal{X}(\mathcal{U}_{R_i}(N))$ such that $\mathcal{P}^{R_i}(Y)$ and $\mu\beta^{R_i}(m_Y) = m_X$.*

Proof. 1. $\mathcal{U}_{R_i}(N)$ is clearly an occurrence net by construction (each transition t can fire just once, consuming the token in the place $(t, *)$, which is initially marked and has no incoming arcs). It is an unravel net as well. Consider a state $X \in \mathcal{X}(\mathcal{U}_{R_i}(N))$ such that $\mathcal{P}^{R_i}(X)$ and assume that $\mathcal{U}_{R_i}(N)|_{[\![X]\!]}$ is not acyclic. Then there must be a place, say (s, i), which is *marked* twice. Assume that ${}^{\bullet}s = \{x, y\}$. X must contain two transitions, $(\!|w|\!)_{\sim}a$ and $(\!|w'|\!)_{\sim}b$, with $a, b \in \{x, y\}$, putting the token in (s, i). As $\mathcal{P}^{R_i}(X)$ it must be that either $(\!|w|\!)_{\sim}a \lhd^* (\!|w'|\!)_{\sim}b$ or $(\!|w'|\!)_{\sim}b \lhd^* (\!|w|\!)_{\sim}a$. Assume the former. This means that (s, i) has been produced for the first time by $(\!|w|\!)_{\sim}a$, but by definition of $\mathcal{U}_{R_i}(N)$ it cannot be produced by $(\!|w'|\!)_{\sim}b$, as the number or elements in ${}^{\bullet}s$ in $(\!|w'|\!)_{\sim}$ is certainly different from those in $(\!|w|\!)_{\sim}$, due to the fact that $(\!|w|\!)_{\sim}a \lhd^* (\!|w'|\!)_{\sim}b$.

Hence $\mathcal{U}_{R_i}(N)|_{[\![X]\!]}$ is acyclic, and is therefore a causal net,
2. $\langle \eta^{R_i}, \beta^{R_i} \rangle : \mathcal{U}_{R_i}(N) \to N$ is a well defined net morphism. Indeed it is routine to check that $\mu\beta^{R_i}(m^{R_i}) = m$, ${}^\bullet\eta^{R_i}((\!|w|\!)_\sim t) = \mu\beta^{R_i}({}^\bullet(\!|w|\!)_\sim t)$ and $\eta^{R_i}((\!|w|\!)_\sim t)^\bullet = \mu\beta^{R_i}((\!|w|\!)_\sim t^\bullet)$, and
3. let $X \in \mathcal{X}(N)$. Consider the firing sequence $m\,[t_1\rangle\, m_1 \ldots [t_n\rangle\, m_X$. We prove that there exists a state $Y \in \mathcal{X}(\mathcal{U}_{R_i}(N))$ such that $\mathcal{P}^{R_i}(Y)$, $\mu\beta^{R_i}(m_Y) = m_X$, and there exists a total order on the elements in Y. The proof is by induction on the length of this firing sequence leading to m_X. If $X = \emptyset$ then $Y = \emptyset$. Assume it holds for $n-1$. Then $X' = X - t_n$ is a state and by induction there is a state $Y' \in \mathcal{X}(\mathcal{U}_{R_i}(N))$ such that $\mu\beta^{R_i}(m_{Y'}) = m_{X'}$ and $\mathcal{P}^{R_i}(Y')$. Take t_n and ${}^\bullet t_n$. Consider the trace $\Downarrow_{\lhd^*} [\![Y']\!] = t_1 \cdots t_{n-1}$ and $s \in {}^\bullet t_n$. The place s is marked at $m_{X'}$ hence there is a place, say (s,i) which is marked at $m_{Y'}$ and $\beta^{R_i}((s,i)) = s$. Clearly $(s,i)^\bullet = \emptyset$ in the subnet $\mathcal{U}_{R_i}(N)|_{[\![Y']\!]}$. Take $t = (\!|\, \|t_1 \cdots t_{n-1}\|_{\circlearrowleft(t_n)} |\!)_\sim \cdot t_n$, make it greater of all the transitions in Y', and consider $Y = Y' + t$. Clearly Y is a state of $\mathcal{U}_{R_i}(N)$, it holds that $\mathcal{P}^{R_i}(Y)$ and $\mu\beta^{R_i}(m_Y) = m_X$.

R_c-unfolding: We can now introduce the second of the two new notions of unfolding of a net, which will turn out again to be an unravel net, and which will denote with $\mathcal{U}_{R_c}$. In the I-unfolding each event codes information on its complete *individual* history, whereas in the R_i-unfolding an event codes all the histories with the same Parikh vector with respect to the neighborhood of the corresponding transition. In this unfolding each event represents the i-th occurrence of the corresponding transition, equating also *alternative* histories.

The places of this unfolding are the same of the previous one, thus the only problem is to identify the transitions.

Consider the unfolding in Fig. 4. The two transitions ba and ca correspond to the same occurrence of the transition a in the original net, as the two past histories can be considered exactly the same from the point of view of the produced effects (i.e., the token in q). We recall that, with respect to the situation depicted in Fig. 3, the i-th occurrence of b consumes a token produced by x or y, and the i-th occurrence of x produces the j-th token in the place s when the number of happening of y, a, b and c are in a precise relation, which we have discussed before.

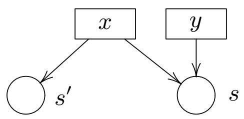

Fig. 5. Part of a safe net

Consider now the situation depicted in Fig. 5. The transition x puts tokens in places s and s', whereas the transition y just put tokens in the place s. The i-th firing of the transition x puts the j-th token in place s and the k-th token in the place s', and in general $k \neq j$. In the R_i-unfolding this was easily accomplished just by counting the occurrences of x and y in the preset of s and of s'. In the R_c-unfolding we indicate explicitly in which places the j-th and k-th tokens are produced. Summing up, an event must have the information on the number of its occurrence and on which occurrence of token is producing in a place. Given a safe net $N = \langle S, T, F, m \rangle$, we assume that S is suitably ordered and with $|S|$

we indicate its cardinality. Transition are then triples $(t, i, \alpha) \in T \times \mathbb{N}^+ \times (\mathbb{N})^{|S|}$, where i indicates the number of occurrences and α is a tuple where the positions corresponding to the places in the postset of t have the information on the token produced in these places. Transitions and places are connected as previously: the transition (t, i, α) can consume the token in place (s, k), provided that $k \leq i$, and it produces a token in the place (s', j), with $\alpha(s') = j$, assuming that $F_{pre}(t, s)$ and $F_{post}(t, s')$. Concerning the property, we simply require that if the i-th occurrence of a transition t is in a state, then also the previous $i - 1$ occurrences must be present, and that the tuples α can be suitably ordered. Given a sequence $\overline{\alpha} = \alpha_1, \ldots, \alpha_n$, with $\overline{\alpha}(s)$ we indicate the sequence of numbers $\alpha_1(s), \ldots, \alpha_n(s)$. We say that $\overline{\alpha} = \alpha_1, \ldots, \alpha_n$ is *complete* if each sequence of numbers obtained by $\overline{\alpha}(s)$ deleting all zeros, denoted with $\mathfrak{s}(\overline{\alpha}(s))$, is either empty, strictly increasing and without gaps, where the latter means that

- $\mathfrak{s}(\overline{\alpha}(s))_i + 1 = \mathfrak{s}(\overline{\alpha}(s))_{i+1}$,
- $\mathfrak{s}(\overline{\alpha}(s))_1 = 0$ if the place is initially unmarked in the original net, and
- $\mathfrak{s}(\overline{\alpha}(s))_1 = 1$ if the place is initially marked in the original net.

Finally let $w \in (T \times \mathbb{N}^+ \times (\mathbb{N})^{|S|})^*$, with α_w we denote the sequence of α associated to w.

Definition 11. *Let $N = \langle S, T, F, m\rangle$ be a safe net. The R_c-unfolding $\mathcal{U}_{R_c}(N) = (\langle S^{R_c}, T^{R_c}, F^{R_c}, m^{R_c}\rangle, \mathcal{P}^{R_c})$ is the net defined as follows:*

$$S^{R_c} = (S \times \mathbb{N}^+) \cup (T^{R_c} \times \{*\})$$

$$T^{R_c} = T \times \mathbb{N}^+ \times (\mathbb{N})^{|S|}$$

$$F^{R_c} = \begin{cases} F^{R_c}_{pre}((t, i, \alpha), (s, k)) \text{ iff } F_{pre}(t, s),\ i \leq k \text{and } k = \alpha(s) - 1 \\ F^{R_c}_{pre}((t, i, \alpha), ((t, i, \alpha), *)) \\ F^{R_c}_{post}((t, i, \alpha), (s, k)) \text{ iff } F_{post}(t, s) \text{ and } k = \alpha(s) + m^{R_c}(s, 1) \end{cases}$$

$$m^{R_c} = \{((s, 1)) \mid m(s) > 0\} \bigcup \{(e, *) \mid e \in T^{R_c}\}$$

$\mathcal{P}^{R_c}(X)$ iff X is a state and for all $(t, i, \alpha) \in [\![X]\!]$, if $i > 1$ then $(t, n, \alpha') \in [\![X]\!]$ for all $n < i$, and let a trace w leading to m_X, the sequence of $\mathfrak{s}(\overline{\alpha_w}(s))$ is complete.

Furthermore $\eta^{R_c} : T^{R_c} \to T$ is defined as $\eta((t, i, \alpha)) = t$ and $\beta^{R_c} : S^{R_c} \to S$ is a multirelation defined as $\beta^{R_c}((s, j)) = s$ iff $s \in S$.

Differently from the R_i-unfolding, the transitions representing the firing of the same transition of the original net and with the same preset and postset in the R_i-unfolding, are now identified. With respect to the previous notion, the two transitions *ca* and *ba*, corresponding to the first occurrence of the transition a in the net N, are now identified in the unfolding in Fig. 6. Observe that these two transitions, in the unfolding in Fig. 4, have the same preset and postset. We can prove a result analogous to the proposition 5 simply by observing that the transitions that are now glued have the same preset and postset in the R_i-unfolding.

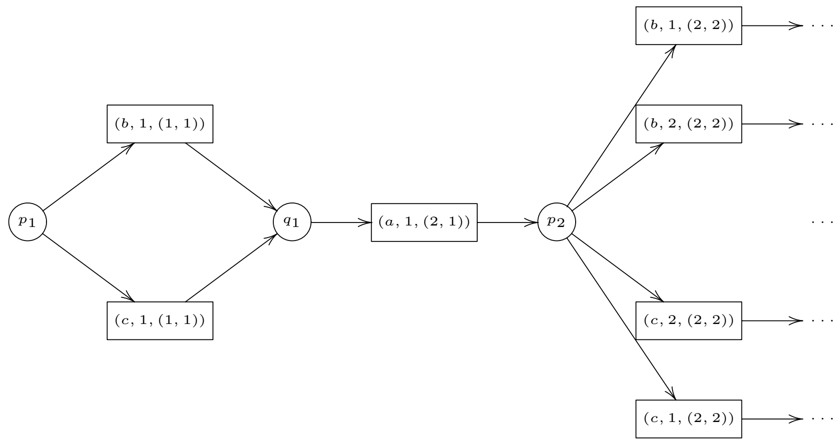

Fig. 6. Part of the R_c-unfolding of N (places $(t,*)$ are omitted).

Proposition 6. *Let $N = \langle S, T, F, m\rangle$ be a safe net and $\mathcal{U}_{R_c}(N) = (\langle S^{R_c}, T^{R_c}, F^{R_c}, m^{R_c}\rangle, \mathcal{P}^{R_c})$ its R_c-unfolding. Then*

1. *$\mathcal{U}_{R_c}(N)$ is an unravel net,*
2. *$\langle \eta^{R_c}, \beta^{R_c}\rangle : \mathcal{U}_{R_c}(N) \to N$ is a well defined net morphism, and*
3. *for each state $X \in \mathcal{X}(N)$ there exists a state $Y \in \mathcal{X}(\mathcal{U}_{R_c}(N))$ such that $\mathcal{P}R_c(Y)$ and $\mu\beta^{R_i}(m_Y) = m_X$.*

Proof. $\mathcal{U}_{R_c}(N)$ is clearly an unravel net, and $\langle \eta^{R_c}, \beta^{R_c}\rangle : \mathcal{U}_{R_c}(N) \to N$ is a well defined net morphism.

Assume the X is a state in $\mathcal{X}(N)$, and consider a firing sequence associated to it. If $X = \emptyset$ then $Y = \emptyset$ is a well defined state. Take $m\,[t_1\rangle\, m_1 \cdots m_{X-t_n}\,[t_n\rangle\, m_x$. By induction to $X - t_n$ a state Y' corresponds and to Y' the firing sequence $m^{R_c}\,[(t_1, 1, \alpha)\rangle\, \hat{m}_1 \ldots m_{Y'}$. Consider $\hat{m}_{i-1}\,[(t_i, k, \alpha_i)\rangle\, \hat{m}_i$, and take $\alpha_i(s) = k$ and $k \neq 0$. Then $\hat{m}_i(s, k)$ is marked. Clearly (t_n, k, α_n) where $k - 1$ is the number of occurrences of t_n in the firing sequence associated to X, and $\alpha_n(s) = l$ if either $s \notin t^\bullet$ and $m_{Y'}(s, i)$ or the place (s, l) does not belong to $\mathcal{U}_{R_c}(N)|_{[\![Y']\!]}$ but $(s, l - 1)$ does, is the transition we can add to Y' obtaining the required Y. Obviously Y satisfies the property $\mathcal{P}^{R_c}$.

Collective unfolding: In the case of the collective token philosophy we adopt a slightly different notion, as we associate to the unfolding also a folding morphism.

Definition 12. *Let $N = \langle S, T, F, m\rangle$ be a safe net. The C-unfolding $\mathcal{U}_C(N) = \langle S^C, T^C, F^C, m^C\rangle$ of the net $N = \langle S, T, F, m\rangle$ is the net defined as follows:*

$$S^C = S \cup (T^C \times \{*\})$$
$$T^C = T \times \mathbb{N}^+$$

$$F^C = F^C_{pre}((t,i),s) \ \textit{if} \ F_{pre}(t,s) \ \textit{or} \ s = ((t,i),*)$$
$$\quad F^C_{post}((t,i),s) \ \textit{if} \ F_{post}(t,s)$$
$$m^C = \{s \mid m(s) > 0\} \bigcup \{(e,*) \mid e \in T^C \ \textit{and} i \in \mathbb{N}^+\}$$

Furthermore $\eta^C : T^C \to T$ *is defined as* $\eta(t,j) = t$ *and* $\beta^C : S^C \to S$ *is a multirelation defined as* $\beta(s) = s$ *iff* $s \in S$

The intuition behind this construction is quite simple: we introduce as many copies of the same transition (which are all distinct because of the index) and we guarantee that they can fire just once. The other connections are just inherited by the original net. Clearly the whole history is lost.

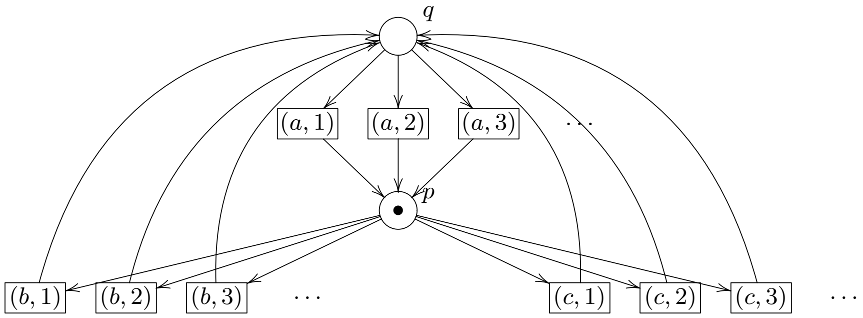

Fig. 7. Part of the C-unfolding of N (places $(t,*)$ are omitted).

Proposition 7. *Let* N *be a safe net and* $\mathcal{U}_C(N)$ *its* C-unfolding. *Then* $\mathcal{U}_C(N)$ *is an occurrence net and* $\langle \eta^C, \beta^C \rangle : \mathcal{U}_C(N) \to N$ *is a well defined net morphism.*

It is straightforward to observe that to a state $X \in \mathcal{X}(N)$ can be easily associated a state Y in $\mathcal{X}(\mathcal{U}_C(N))$, by simply adding different indexes to occurrences of the transitions in X.

4 Relating the notions of unfolding

In this section we relate the notions introduced so far. We first show how the I and the R_i-unfoldings are related. Let $N = \langle S, T, F, m \rangle$ be a safe net and $\mathcal{U}^I(N) = \langle B, E, F, m \rangle$ be its I-unfolding. With $\lceil (Y,s) \rceil_s$ we denote the number of conditions (Y', s) *preceedings* the condition (Y,s) in $\mathcal{U}^I(N)$ (including (Y,s)). Consider now an event (X,t) and a marking m' in $\mathcal{U}^I(N)$ such that $m' \, [(X,t)\rangle$. To m' corresponds the marking $\beta^{R_i}(m')$, and in particular a firing sequence. A trace z leading to (X,t) is then obtained by this firing sequence. Though there can be more then one, as we are interested in the transitions in the neighborhood of t, all the traces can be considered as equivalent, as soon as they have the same number of occurrences of the transitions in the neighborhood of the transition corresponding to the event (X,t).

Theorem 1. *Let* $N = \langle S, T, F, m \rangle$ *be a safe net. Then there exists a folding morphism* $\langle \eta, \beta \rangle : \mathcal{U}^I(N) \to \mathcal{U}^{R_i}(N)$.

Proof. We construct the morphism as follows: $\beta((Y,s),(s,i))$ iff $\lceil (Y,s) \rceil_s = i$ and $\beta((\{e\},*),(\eta(e),*))$, and $\eta(X,t) = (\!| w |\!)_{\sim} t$ where $w = \|z\|_{\circlearrowleft(t)}$ and z is a trace leading to (X,t) in $\mathcal{U}^I(N)$. It is routine to check that $\langle \eta, \beta \rangle$ is a well defined net morphism.

The second result is the one relating the two new unfoldings we have introduced.

Theorem 2. *Let $N = \langle S, T, F, m \rangle$ be a safe net. Then there exists a folding morphism $\langle \eta, \beta \rangle : \mathcal{U}^{R_i}(N) \to \mathcal{U}^{R_c}(N)$.*

Proof. We construct the morphism as follows: β is the identity multirelation on places (s,i) and and $\beta((\{e\},*),(\eta(e),*))$ otherwise, and $\eta((\!| w |\!)_{\sim} t) = (t,i,\alpha)$ where $i = [w]_t$ and $\alpha(s) = [w]_{\bullet s}$. Clearly $\mu\beta(\,^{\bullet}(\!| w |\!)_{\sim} t) = \,^{\bullet}\eta((\!| w |\!)_{\sim} t)$ and $\mu\beta((\!| w |\!)_{\sim} t^{\bullet}) = \eta((\!| w |\!)_{\sim} t)^{\bullet}$.

Finally we relate the R_c-unfolding to the C-unfolding.

Theorem 3. *Let $N = \langle S, T, F, m \rangle$ be a safe net. Then there exists a folding morphism $\langle \eta, \beta \rangle : \mathcal{U}^{R_c}(N) \to \mathcal{U}^C(N)$.*

Proof. We construct the morphism as follows: $\beta((s,i),s)$, $\beta((\{e\},*),(\eta(e),*))$, and $\eta((t,i,\alpha)) = (t,i)$. It is clearly a well defined morphism.

The relation between the collective unfolding of a net and the individual one is more subtle, as we have to guess a history to be able to relate the collective unfolding to the individual one. If we consider a state of the collective unfolding, we can *unfold* it onto the individual one. Thus we can fix an order in the execution of different occurrences of the same transition (which is arbitrary) and also keep track of the dependencies among different transitions by exploiting which tokens are produced and consumed (for a proof, see [14]).

Theorem 4. *Let $N = \langle S, T, F, m \rangle$ be a Petri net and let $\mathcal{U}_I(N) = \langle B^I, E^I, F^I, m^I \rangle$ and $\mathcal{U}_C(N) = \langle S^C, T^C, F^C, m^C \rangle$ be the individual (resp. collective) unfolding of N. Let X be a state of $\mathcal{U}_C(N)$ and m_X be the reached marking. Then there exists a finite subnet of $N' = \langle B', E', F', m' \rangle$ of $\mathcal{U}_I(N)$ such that*

1. *each reachable marking of N' is a marking of a firing sequence leading to m_X, and*
2. *there exists an one to one correspondence between E' and X.*

Furthermore N' is a causal net such that $|b^{\bullet}| \leq 1$ for all $b \in B'$.

5 How much of the history can be forgotten?

Before discussing the results presented in this paper, we compare our constructions with the one introduced in [11].

The occurrence depth of a condition c in C is the maximum number of elements with the same label that are on a path from an initial condition to c (a path is just the set of totally ordered elements less then c). A merged process of a safe net is defined as follows:

Definition 13. *Let $\mathcal{U}_I(N) = \langle B^I, E^I, F^I, m^I\rangle$ be the I-unfolding of the net $N = \langle S, T, F, m\rangle$. Then $\mathfrak{M}(\mathcal{U}_I(N))$ is the unravel net obtained with the following two steps:*

Step 1: *the places of $\mathfrak{M}(\mathcal{U}_I(N))$ are obtained by fusing together all the conditions of $\mathcal{U}_I(N)$ which have the same labels and occurrence-depths; each* mp-*condition inherits its label and arcs from the fused conditions, and its initial marking is the total number of minimal conditions which were fused into it, and*

Step 2: *the transitions of $\mathfrak{M}(\mathcal{U}_I(N))$, called* mp-*events, are obtained by merging all the events which have the same labels, presets and postsets (after step 1 was performed); each mp-event inherits its label from the merged events (and has exactly the same connectivity as either of them).*

Certainly both our unfoldings represent places in the same way, but there are differences in the transitions, as the merged process may identify transitions corresponding to different occurrences of the same transitions. Consider the unfolding in Fig. 2 and apply to the causal net the construction of definition 13. We obtain the following: where the two occurrence of b after the first a are iden-

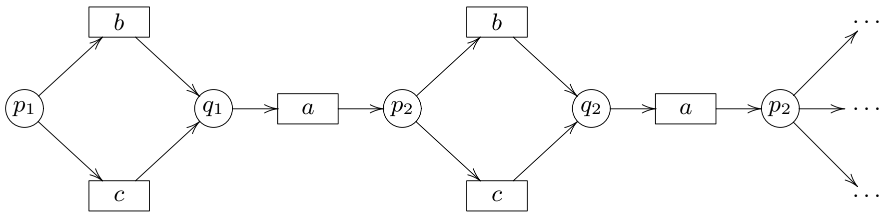

Fig. 8. Part of the merged process of N (places $(t, *)$ are omitted).

tified. Thus a merged process is a more compact representation of the unfolding, though retrieving which is actually the correct run of the system may be less direct with respect to the approach taken here. The information which is abstracted away in the merged processes view with respect to the R_c-unfolding is the *index* of the occurrence of the transition, thus there exists a mapping from the R_c-unfolding to the merged process of a net N. However this relation does not fit in the view of this paper as there is no obvious relation between merged processes and collective unfolding. In fact our target was different with respect to the one of [11], as we have focussed our attention on how much of the history can be kept by direct constructions. To this aim we have introduced two new notions of unfolding, obtaining a more compact and possibly useful representation of the non sequential behaviour of a safe net.

Let us sum up the results. The taxonomy emerging from these notions is the following:

- the individual unfolding models the whole history of each token and happening of a transition. As the whole history is kept, the same information is spread in many points of the unfolding, and this information could be represented more compactly;
- the first step toward a more compact representation, still being able to reconstruct easily the whole history, is represented by the R_i-unfolding, where the i-th occurrences of a token in a place are equated. In this unfolding a transition keeps its local history, namely the counts of the occurrences of transitions in its neighborhood;
- the second step is to identify the happening of the transitions that, though may have different histories, are from the point of view of external observation, undistinguishable: they produce the same tokens and correspond to the same occurrence of the transition (i.e., the case of the i-th firing of a transition). Nevertheless some relevant dependencies are kept, as it is still possible to look at the unfolding and retrieve a part of the history;
- finally the history can be totally forgotten, in the collective unfolding. In this case the only way to find out the dependencies among occurrences of different transitions is to look at all the possible executions.

The relations among these notions show that the *fading* of the history can be modeled. To obtain these results we have formalized a notion of unravel net, where the original net is not completely unfolded, but rather somehow *unraveled*.

Concerning to the question '*How much of the history can be forgotten*', the answer obviously depend on the use of the history one want to realize. Here we have shown that there are construction in between the two extrema of what it seems to be a whole spectrum. The problem of retrieving a net from an unfolding has not be pursued in this paper, but the classical approaches do applies to the unfoldings presented, provided that the property is kept into account.

Though we have not addressed the problem of the unfolding in the case of non safe nets, our constructions can be easily lifted to take into account these nets, observing that the relations among indexes are much more complex. Furthermore the property characterizing unravel nets may be presented using suitable combinations of relations which can be extracted from the net itself, as it is done in causal net.

Many relevant issues have not been addressed in this paper. We just point out two of them. One concerns the relation of these unfoldings with event structures, and the other regards the categorical investigation of these constructions. These will be the subject of further investigations.

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